EXERCISE 8.4

1. If cosecθ + cotθ = p, then prove that cosθ = (p² – 1)/ (p² + 1).

Solution:

According to the question,

cosec θ + cot θ = p

Since,

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} \ \&\ \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = p$$

$$\frac{1+\cos\theta}{\sin\theta} = p$$

Squaring on L.H.S and R.H.S,

$$\left(\frac{1+\cos\theta}{\sin\theta}\right)^2 = p^2$$

$$\frac{1+\cos^2\theta+2\cos\theta}{\sin^2\theta} = p^2$$

Applying component and dividend rule,

$$\frac{(1+\cos^2\theta+2\cos\theta)-\sin^2\theta}{(1+\cos^2\theta+2\cos\theta)+\sin^2\theta} = \frac{p^2-1}{p^2+1}$$

$$\frac{(1-\sin^2\theta)+\cos^2\theta+2\cos\theta}{\sin^2\theta+\cos^2\theta+1+2\cos\theta} = \frac{p^2-1}{p^2+1}$$

Since,

$$1-\sin^2\theta = \cos^2\theta \ \&\ \sin^2\theta+\cos^2\theta = 1$$

$$\frac{\cos^2\theta+\cos^2\theta+2\cos\theta}{1+1+2\cos\theta} = \frac{p^2-1}{p^2+1}$$

$$\frac{2\cos^2\theta+2\cos\theta}{2+2\cos\theta} = \frac{p^2-1}{p^2+1}$$

$$\frac{2\cos\theta\,(\cos\theta+1)}{2\,(\cos\theta+1)} = \frac{p^2-1}{p^2+1}$$

$$\cos\theta = \frac{p^2-1}{p^2+1}$$

Hence, proved.

2. Prove that √(sec² θ + cosec² θ) = tan θ + cot θ

Solution:

L.H.S=

$\sqrt{(\sec^2\theta + \text{cosec}^2\theta)}$

Since,

$\sec^2\theta = \frac{1}{\cos^2\theta}$ & $\csc^2\theta = \frac{1}{\sin^2\theta}$

$= \sqrt{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}}$

$= \sqrt{\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}}$

Since,

$\sin^2\theta + \cos^2\theta = 1$

$= \sqrt{\frac{1}{\cos^2\theta\sin^2\theta}}$

$= \frac{1}{\cos\theta\sin\theta}$

Since,

$1 = \sin^2\theta + \cos^2\theta$

$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{\sin^2\theta}{\cos\theta\sin\theta} + \frac{\cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

Since,

$\frac{\sin\theta}{\cos\theta} = \tan\theta$ & $\frac{\cos\theta}{\sin\theta} = \cot\theta$

$= \tan\theta + \cot\theta$

= R.H.S

Hence, proved.

3. The angle of elevation of the top of a tower from certain point is 30°. If the observer moves 20 metres towards the tower, the angle of elevation of the top increases by 15°. Find the height of the tower.

Solution:

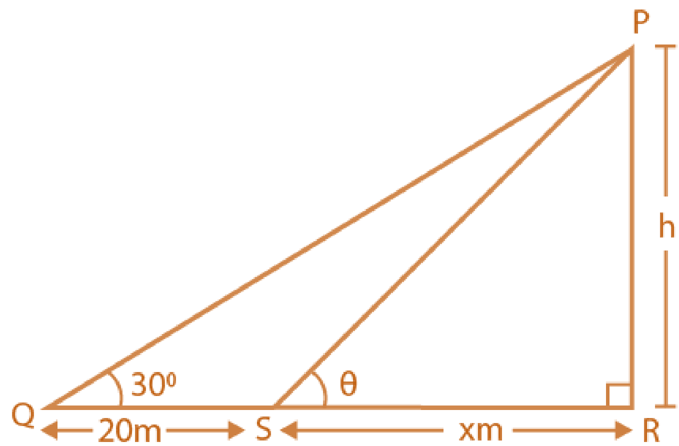
Let PR = h meter, be the height of the tower.

The observer is standing at point Q such that, the distance between the observer and tower is QR = (20+x) m, where

QR = QS + SR = 20 + x

∠PQR = 30°

∠ PSR = θ



In ΔPQR,

$\tan 30° = \frac{h}{20+x}$ [∵, $\tan\theta = \frac{\text{perpendicular}}{\text{base}}$]

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{20+x}$ [∵, $\tan 30° = \frac{1}{\sqrt{3}}$]

Rearranging the terms,
We get $20 + x = \sqrt{3}h$
$\Rightarrow x = \sqrt{3}h - 20$ …eq.1
In ΔPSR,
$\tan\theta = h/x$
Since, angle of elevation increases by 15° when the observer moves 20 m towards the tower.
We have,
$\theta = 30° + 15° = 45°$
So,
$\tan 45° = h/x$
$\Rightarrow 1 = h/x$
$\Rightarrow h = x$
Substituting x=h in eq. 1, we get
$h = \sqrt{3}\,h - 20$
$\Rightarrow \sqrt{3}\,h - h = 20$
$\Rightarrow h(\sqrt{3} - 1) = 20$

$\Rightarrow h = \frac{20}{\sqrt{3}-1}$

Rationalizing the denominator, we have

$\Rightarrow h = \frac{20}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$

$\Rightarrow h = \frac{20(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$= \frac{20(\sqrt{3}+1)}{3-1}$

$= \frac{20(\sqrt{3}+1)}{2}$

$= 10(\sqrt{3} + 1)$

Hence, the required height of the tower is $10(\sqrt{3} + 1)$ meter.

4. If $1 + \sin^2\theta = 3\sin\theta \cos\theta$, then prove that $\tan\theta = 1$ or ½.

Solution:

Given: $1+\sin^2 \theta = 3 \sin \theta \cos \theta$

Dividing L.H.S and R.H.S equations with $\sin^2 \theta$,

We get,

$$\frac{1+\sin^2\theta}{\sin^2\theta} = \frac{3\sin\theta\cos\theta}{\sin^2\theta}$$

$$\Rightarrow \frac{1}{\sin^2\theta} + 1 = \frac{3\cos\theta}{\sin\theta}$$

$\mathrm{cosec}^2 \theta + 1 = 3 \cot \theta$

Since,

$\mathrm{cosec}^2 \theta - \cot^2 \theta = 1 \Rightarrow \mathrm{cosec}^2 \theta = \cot^2 \theta +1$

$\Rightarrow \cot^2 \theta +1+1 = 3 \cot \theta$

$\Rightarrow \cot^2 \theta +2 = 3 \cot \theta$

$\Rightarrow \cot^2 \theta -3 \cot \theta +2 = 0$

Splitting the middle term and then solving the equation,

$\Rightarrow \cot^2 \theta - \cot \theta -2 \cot \theta +2 = 0$

$\Rightarrow \cot \theta(\cot \theta -1)-2(\cot \theta +1) = 0$

$\Rightarrow (\cot \theta - 1)(\cot \theta - 2) = 0$

$\Rightarrow \cot \theta = 1, 2$

Since,

$\tan \theta = 1/\cot \theta$

$\tan \theta = 1$, ½

Hence, proved.

5. Given that $\sin\theta + 2\cos\theta = 1$, then prove that $2\sin\theta - \cos\theta = 2$.

Solution:

Given: $\sin \theta +2 \cos \theta = 1$

Squaring on both sides,

$(\sin \theta +2 \cos \theta)^2 = 1$

$\Rightarrow \sin^2 \theta + 4 \cos^2 \theta + 4\sin \theta\cos \theta = 1$

Since, $\sin^2 \theta = 1 - \cos^2 \theta$ and $\cos^2 \theta = 1 - \sin^2 \theta$

$\Rightarrow (1 - \cos^2 \theta) + 4(1 - \sin^2 \theta) + 4\sin \theta\cos \theta = 1$

$\Rightarrow 1 - \cos^2 \theta + 4 - 4 \sin^2 \theta + 4\sin \theta\cos \theta = 1$

$\Rightarrow - 4 \sin^2 \theta - \cos^2 \theta + 4\sin \theta\cos \theta = - 4$

$\Rightarrow 4 \sin^2 \theta + \cos^2 \theta - 4\sin \theta\cos \theta = 4$

We know that,

$a^2+ b^2 - 2ab = (a - b)^2$

So, we get,

$(2\sin \theta - \cos \theta)^2 = 4$

$\Rightarrow 2\sin \theta - \cos \theta = 2$

Hence proved.

6. The angle of elevation of the top of a tower from two points distant s and t from its foot are complementary. Prove that the height of the tower is $\sqrt{st}$.

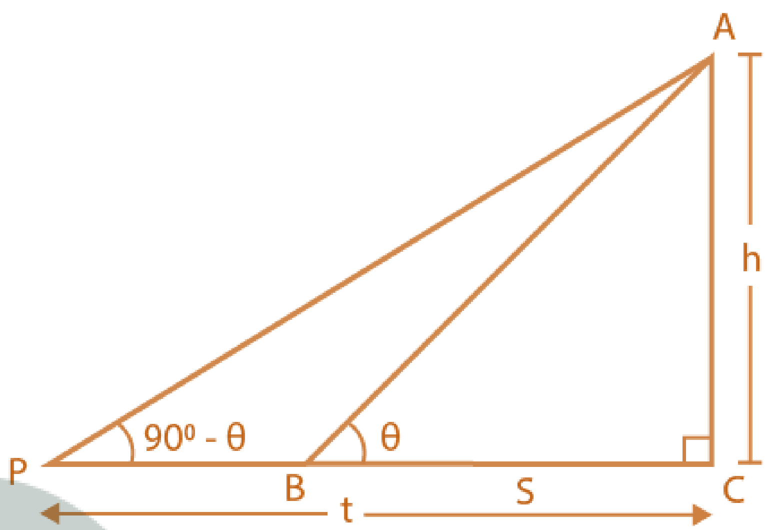
Solution:

Let BC = s; PC = t

Let height of the tower be AB = h.

∠ABC = θ and ∠APC = 90° - θ

(∵ the angle of elevation of the top of the tower from two points P and B are complementary)



In $\Delta ABC, \tan\theta = \frac{AC}{BC} = \frac{h}{s}$ ... eq. 1 $[\because, \tan\theta = \frac{\text{perpendicular}}{\text{base}}]$

In $\Delta APC, \tan(90° - \theta) = \frac{AC}{PC} = \frac{h}{t}$

$\Rightarrow \cot\theta = \frac{h}{t}$ ... eq. 2

Multiplying eq. 1 and eq. 2, we get

$\tan\theta \times \cot\theta = \frac{h}{s} \times \frac{h}{t}$

$\Rightarrow 1 = \frac{h^2}{st}$ $[\because \tan\theta \times \cot\theta = 1 \text{ as } \cot\theta = \frac{1}{\tan\theta}]$

$\Rightarrow h^2 = st$

$\Rightarrow h = \sqrt{st}$

Hence the height of the tower is $\sqrt{st}$.

7. The shadow of a tower standing on a level plane is found to be 50 m longer when Sun's elevation is 30° than when it is 60°. Find the height of the tower.

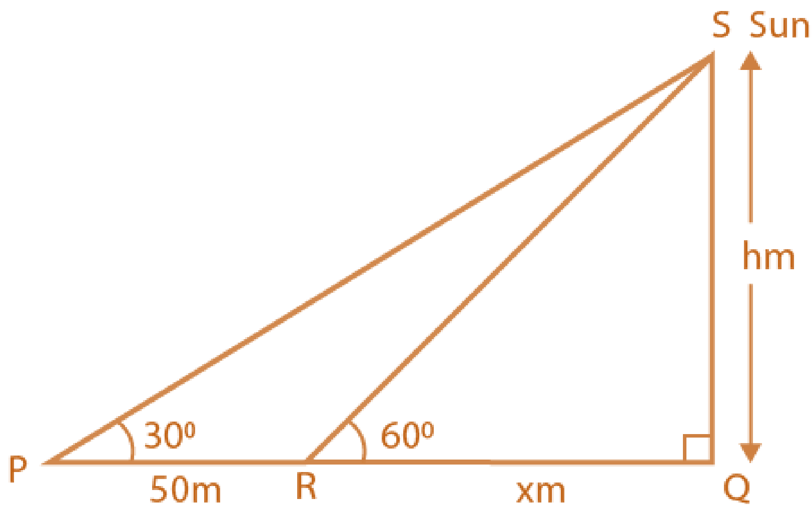
Solution:

Let SQ = h be the tower.

∠SPQ = 30° and ∠SRQ = 60°

According to the question, the length of shadow is 50 m long hen angle of elevation of the sun is 30° than when it was 60°. So,

PR = 50 m and RQ = x m



So in ΔSRQ, we have

$\tan 60° = \frac{h}{x}$

$[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}$

$\Rightarrow \tan 60° = \frac{SQ}{RQ}]$

$\Rightarrow \sqrt{3} = \frac{h}{x}\ [\because \tan 60° = \sqrt{3}]$

$\Rightarrow x = \frac{h}{\sqrt{3}}$

In ΔSPQ,

$\tan 30° = \frac{h}{50+x}$

$[\because \tan 30° = \frac{SQ}{PQ} = \frac{SQ}{PR+PQ}]$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{50+x}\ [\because, \tan 30° = \frac{1}{\sqrt{3}}]$

$\Rightarrow 50 + x = \sqrt{3}h$

Substituting the value of x in the above equation, we get

$\Rightarrow 50 + \frac{h}{\sqrt{3}} = \sqrt{3}h$

$\Rightarrow \frac{50\sqrt{3}+h}{\sqrt{3}} = \sqrt{3}h$

$\Rightarrow 50\sqrt{3}+h = 3h$

$\Rightarrow 50\sqrt{3} = 3h - h$

$\Rightarrow 3h - h = 50\sqrt{3}$

$\Rightarrow 2h = 50\sqrt{3}$

$\Rightarrow h = (50\sqrt{3})/2$

$\Rightarrow h = 25\sqrt{3}$

Hence, the required height is $25\sqrt{3}$ m.

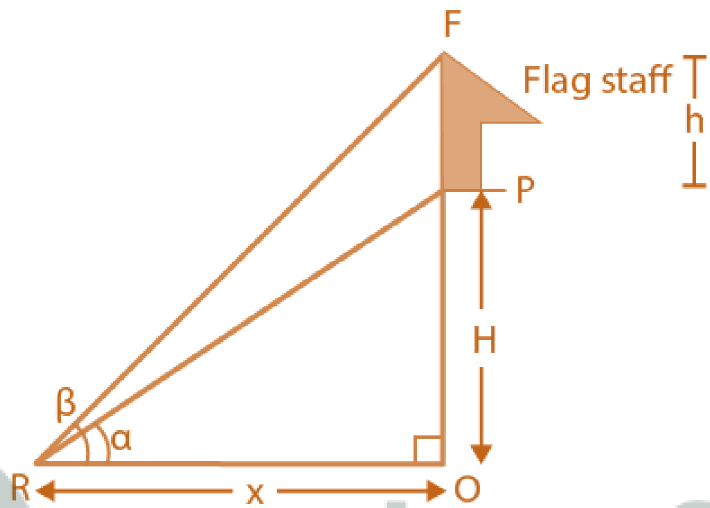
8. A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height

h. At a point on the plane, the angles of elevation of the bottom and the top of the flag staff are α and β, respectively. Prove that the height of the tower is [h tan α/(tan β – tan α)].

Solution:

Given that a vertical flag staff of height h is surmounted on a vertical tower of height H(say), such that FP = h and FO = H.

The angle of elevation of the bottom and top of the flag staff on the plane is ∠PRO = α and ∠FRO = β respectively



In ΔPRO, we have

$$\tan\alpha = \frac{PO}{RO} = \frac{H}{x}$$

$$[\because \tan\theta = \frac{\text{perpendicular}}{\text{base}}]$$

$$\Rightarrow x = \frac{H}{\tan\alpha} \text{ ... eq. 1}$$

And in ΔFRO, we have

$$\tan\beta = \frac{FO}{RO} = \frac{FP+PO}{RO} = \frac{h+H}{x}$$

$$\Rightarrow x = \frac{h+H}{\tan\beta} \text{ ... eq. 2}$$

Comparing eq. 1 and eq. 2,

$$\Rightarrow \frac{H}{\tan\alpha} = \frac{h+H}{\tan\beta}$$

Solving for H,

$\Rightarrow H \tan\beta = (h+H) \tan\alpha$

$\Rightarrow H \tan\beta - H \tan\alpha = h \tan\alpha$

$\Rightarrow H (\tan\beta - \tan\alpha) = h \tan\alpha$

$$\Rightarrow H = \frac{h\tan\alpha}{\tan\beta - \tan\alpha}$$

Hence, proved.

9. If tanθ + secθ = l, then prove that secθ = $(l^2 + 1)/2l$.

Solution:

Given: tan θ+ sec θ = l …eq. 1

Multiplying and dividing by (sec θ – tan θ) on numerator and denominator of L.H.S,

$$\frac{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{\sec\theta-\tan\theta} = l$$

$$\Rightarrow \frac{\sec^2\theta-\tan^2\theta}{\sec\theta-\tan\theta} = l$$

Since, $\sec^2\theta - \tan^2\theta = 1$

$$\Rightarrow \frac{1}{\sec\theta-\tan\theta} = l$$

So, sec θ – tan θ = 1 …eq.2

Adding eq. 1and eq. 2, we get

(tan θ + sec θ) + (sec θ – tan θ) = 1

$$\Rightarrow 2\sec\theta = \frac{l^2+1}{l}$$

$$\Rightarrow \sec\theta = \frac{l^2+1}{2l}$$

Hence, proved.