

EXERCISE 2.2

1. Find $g \circ f$ and $f \circ g$ when $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

(i) $f(x) = 2x + 3$ and $g(x) = x^2 + 5$.

(ii) $f(x) = 2x + x^2$ and $g(x) = x^3$

(iii) $f(x) = x^2 + 8$ and $g(x) = 3x^3 + 1$

(iv) $f(x) = x$ and $g(x) = |x|$

(v) $f(x) = x^2 + 2x - 3$ and $g(x) = 3x - 4$

(vi) $f(x) = 8x^3$ and $g(x) = x^{1/3}$

Solution:

(i) Given, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

So, $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$ and $f \circ g: \mathbb{R} \rightarrow \mathbb{R}$

Also given that $f(x) = 2x + 3$ and $g(x) = x^2 + 5$

Now, $(g \circ f)(x) = g(f(x))$

$$= g(2x + 3)$$

$$= (2x + 3)^2 + 5$$

$$= 4x^2 + 9 + 12x + 5$$

$$= 4x^2 + 12x + 14$$

Now, $(f \circ g)(x) = f(g(x))$

$$= f(x^2 + 5)$$

$$= 2(x^2 + 5) + 3$$

$$= 2x^2 + 10 + 3$$

$$= 2x^2 + 13$$

(ii) Given, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

so, $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$ and $f \circ g: \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = 2x + x^2$ and $g(x) = x^3$

$(g \circ f)(x) = g(f(x))$

$$= g(2x + x^2)$$

$$= (2x + x^2)^3$$

Now, $(f \circ g)(x) = f(g(x))$

$$= f(x^3)$$

$$= 2(x^3) + (x^3)^2$$

$$= 2x^3 + x^6$$

(iii) Given, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

So, $\text{gof}: \mathbb{R} \rightarrow \mathbb{R}$ and $\text{fog}: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^2 + 8 \text{ and } g(x) = 3x^3 + 1$$

$$(\text{gof})(x) = g(f(x))$$

$$= g(x^2 + 8)$$

$$= 3(x^2 + 8)^3 + 1$$

$$\text{Now, } (\text{fog})(x) = f(g(x))$$

$$= f(3x^3 + 1)$$

$$= (3x^3 + 1)^2 + 8$$

$$= 9x^6 + 6x^3 + 1 + 8$$

$$= 9x^6 + 6x^3 + 9$$

(iv) Given, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

So, $\text{gof}: \mathbb{R} \rightarrow \mathbb{R}$ and $\text{fog}: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x \text{ and } g(x) = |x|$$

$$(\text{gof})(x) = g(f(x))$$

$$= g(x)$$

$$= |x|$$

$$\text{Now } (\text{fog})(x) = f(g(x))$$

$$= f(|x|)$$

$$= |x|$$

(v) Given, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

So, $\text{gof}: \mathbb{R} \rightarrow \mathbb{R}$ and $\text{fog}: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^2 + 2x - 3 \text{ and } g(x) = 3x - 4$$

$$(\text{gof})(x) = g(f(x))$$

$$= g(x^2 + 2x - 3)$$

$$= 3(x^2 + 2x - 3) - 4$$

$$= 3x^2 + 6x - 9 - 4$$

$$= 3x^2 + 6x - 13$$

$$\text{Now, } (\text{fog})(x) = f(g(x))$$

$$= f(3x - 4)$$

$$= (3x - 4)^2 + 2(3x - 4) - 3$$

$$= 9x^2 + 16 - 24x + 6x - 8 - 3$$

$$= 9x^2 - 18x + 5$$

(vi) Given, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

So, $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$ and $f \circ g: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = 8x^3 \text{ and } g(x) = x^{1/3}$$

$$(g \circ f)(x) = g(f(x))$$

$$= g(8x^3)$$

$$= (8x^3)^{1/3}$$

$$= [(2x)^3]^{1/3}$$

$$= 2x$$

Now, $(f \circ g)(x) = f(g(x))$

$$= f(x^{1/3})$$

$$= 8(x^{1/3})^3$$

$$= 8x$$

2. Let $f = \{(3, 1), (9, 3), (12, 4)\}$ and $g = \{(1, 3), (3, 3), (4, 9), (5, 9)\}$. Show that $g \circ f$ and $f \circ g$ are both defined. Also, find $f \circ g$ and $g \circ f$.

Solution:

Given $f = \{(3, 1), (9, 3), (12, 4)\}$ and $g = \{(1, 3), (3, 3), (4, 9), (5, 9)\}$

$f: \{3, 9, 12\} \rightarrow \{1, 3, 4\}$ and $g: \{1, 3, 4, 5\} \rightarrow \{3, 9\}$

Co-domain of f is a subset of the domain of g .

So, $g \circ f$ exists and $g \circ f: \{3, 9, 12\} \rightarrow \{3, 9\}$

$$(g \circ f)(3) = g(f(3)) = g(1) = 3$$

$$(g \circ f)(9) = g(f(9)) = g(3) = 3$$

$$(g \circ f)(12) = g(f(12)) = g(4) = 9$$

$$\Rightarrow g \circ f = \{(3, 3), (9, 3), (12, 9)\}$$

Co-domain of g is a subset of the domain of f .

So, $f \circ g$ exists and $f \circ g: \{1, 3, 4, 5\} \rightarrow \{3, 9, 12\}$

$$(f \circ g)(1) = f(g(1)) = f(3) = 1$$

$$(f \circ g)(3) = f(g(3)) = f(3) = 1$$

$$(f \circ g)(4) = f(g(4)) = f(9) = 3$$

$$(f \circ g)(5) = f(g(5)) = f(9) = 3$$

$$\Rightarrow f \circ g = \{(1, 1), (3, 1), (4, 3), (5, 3)\}$$

3. Let $f = \{(1, -1), (4, -2), (9, -3), (16, 4)\}$ and $g = \{(-1, -2), (-2, -4), (-3, -6), (4, 8)\}$. Show that $g \circ f$ is defined while $f \circ g$ is not defined. Also, find $g \circ f$.

Solution:

Given $f = \{(1, -1), (4, -2), (9, -3), (16, 4)\}$ and $g = \{(-1, -2), (-2, -4), (-3, -6), (4, 8)\}$

$f: \{1, 4, 9, 16\} \rightarrow \{-1, -2, -3, 4\}$ and $g: \{-1, -2, -3, 4\} \rightarrow \{-2, -4, -6, 8\}$

Co-domain of f = domain of g

So, $g \circ f$ exists and $g \circ f: \{1, 4, 9, 16\} \rightarrow \{-2, -4, -6, 8\}$

$$(g \circ f)(1) = g(f(1)) = g(-1) = -2$$

$$(g \circ f)(4) = g(f(4)) = g(-2) = -4$$

$$(g \circ f)(9) = g(f(9)) = g(-3) = -6$$

$$(g \circ f)(16) = g(f(16)) = g(4) = 8$$

So, $g \circ f = \{(1, -2), (4, -4), (9, -6), (16, 8)\}$

But the co-domain of g is not same as the domain of f .

So, $f \circ g$ does not exist.

4. Let $A = \{a, b, c\}$, $B = \{u, v, w\}$ and let f and g be two functions from A to B and from B to A , respectively, defined as: $f = \{(a, v), (b, u), (c, w)\}$, $g = \{(u, b), (v, a), (w, c)\}$. Show that f and g both are bijections and find $f \circ g$ and $g \circ f$.

Solution:

Given $f = \{(a, v), (b, u), (c, w)\}$, $g = \{(u, b), (v, a), (w, c)\}$.

Also given that $A = \{a, b, c\}$, $B = \{u, v, w\}$

Now we have to show f and g both are bijective.

Consider $f = \{(a, v), (b, u), (c, w)\}$ and $f: A \rightarrow B$

Injectivity of f : No two elements of A have the same image in B .

So, f is one-one.

Surjectivity of f : Co-domain of $f = \{u, v, w\}$

Range of $f = \{u, v, w\}$

Both are same.

So, f is onto.

Hence, f is a bijection.

Now consider $g = \{(u, b), (v, a), (w, c)\}$ and $g: B \rightarrow A$

Injectivity of g : No two elements of B have the same image in A .

So, g is one-one.

Surjectivity of g : Co-domain of $g = \{a, b, c\}$

Range of $g = \{a, b, c\}$

Both are the same.

So, g is onto.

Hence, g is a bijection.

Now we have to find $f \circ g$,

we know that Co-domain of g is same as the domain of f .

So, fog exists and fog: $\{u, v, w\} \rightarrow \{u, v, w\}$

$$(fog)(u) = f(g(u)) = f(b) = u$$

$$(fog)(v) = f(g(v)) = f(a) = v$$

$$(fog)(w) = f(g(w)) = f(c) = w$$

So, fog = $\{(u, u), (v, v), (w, w)\}$

Now we have to find gof,

Co-domain of f is same as the domain of g.

So, fog exists and gof: $\{a, b, c\} \rightarrow \{a, b, c\}$

$$(gof)(a) = g(f(a)) = g(v) = a$$

$$(gof)(b) = g(f(b)) = g(u) = b$$

$$(gof)(c) = g(f(c)) = g(w) = c$$

So, gof = $\{(a, a), (b, b), (c, c)\}$

5. Find fog (2) and gof (1) when $f: \mathbb{R} \rightarrow \mathbb{R}; f(x) = x^2 + 8$ and $g: \mathbb{R} \rightarrow \mathbb{R}; g(x) = 3x^3 + 1$.

Solution:

Given $f: \mathbb{R} \rightarrow \mathbb{R}; f(x) = x^2 + 8$ and $g: \mathbb{R} \rightarrow \mathbb{R}; g(x) = 3x^3 + 1$.

Consider $(fog)(2) = f(g(2))$

$$= f(3 \times 2^3 + 1)$$

$$= f(3 \times 8 + 1)$$

$$= f(25)$$

$$= 25^2 + 8$$

$$= 633$$

$(gof)(1) = g(f(1))$

$$= g(1^2 + 8)$$

$$= g(9)$$

$$= 3 \times 9^3 + 1$$

$$= 2188$$

6. Let \mathbb{R}^+ be the set of all non-negative real numbers. If $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are defined as $f(x) = x^2$ and $g(x) = \sqrt{x}$, find fog and gof. Are they equal functions.

Solution:

Given $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$

So, fog: $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ and gof: $\mathbb{R}^+ \rightarrow \mathbb{R}^+$

Domains of fog and gof are the same.

Now we have to find fog and gof also we have to check whether they are equal or not,

$$\begin{aligned}\text{Consider } (f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{x}) \\ &= \sqrt{x^2} \\ &= x\end{aligned}$$

$$\begin{aligned}\text{Now consider } (g \circ f)(x) &= g(f(x)) \\ &= g(x^2) \\ &= \sqrt{x^2} \\ &= x\end{aligned}$$

So, $(f \circ g)(x) = (g \circ f)(x), \forall x \in \mathbb{R}^+$

Hence, $f \circ g = g \circ f$

7. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$ and $g(x) = x + 1$. Show that $f \circ g \neq g \circ f$.

Solution:

Given $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$.

So, the domains of f and g are the same.

$$\begin{aligned}\text{Consider } (f \circ g)(x) &= f(g(x)) \\ &= f(x + 1) = (x + 1)^2 \\ &= x^2 + 1 + 2x\end{aligned}$$

$$\begin{aligned}\text{Again consider } (g \circ f)(x) &= g(f(x)) \\ &= g(x^2) = x^2 + 1\end{aligned}$$

So, $f \circ g \neq g \circ f$

