

$\bar{r} = \bar{a}_2 + \lambda \bar{b}$ is given by,

$$d = \frac{|(\bar{a}_2 - \bar{a}_1) \times \bar{b}|}{|\bar{b}|}$$

Answer :

For given lines,

$$\bar{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\bar{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Here,

$$\bar{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}$$

$$\bar{b}_1 = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\bar{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\bar{b}_2 = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

As $\bar{b}_1 = \bar{b}_2 = \bar{b}$ (say), given lines are parallel to each other.

Therefore,

$$\bar{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\therefore |\bar{b}| = \sqrt{2^2 + 3^2 + 6^2}$$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49}$$

$$= 7$$

$$\bar{a}_2 - \bar{a}_1 = (3 - 1)\hat{i} + (3 - 2)\hat{j} + (-5 + 4)\hat{k}$$

$$\therefore \bar{a}_2 - \bar{a}_1 = 2\hat{i} + \hat{j} - \hat{k}$$

$$(\bar{a}_2 - \bar{a}_1) \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix}$$

$$= \hat{i}(6 + 3) - \hat{j}(12 + 2) + \hat{k}(6 - 2)$$

$$\therefore (\bar{a}_2 - \bar{a}_1) \times \bar{b} = 9\hat{i} - 14\hat{j} + 4\hat{k}$$

$$\therefore |(\bar{a}_2 - \bar{a}_1) \times \bar{b}| = \sqrt{9^2 + (-14)^2 + 4^2}$$

$$= \sqrt{81 + 196 + 16}$$



$$= \sqrt{293}$$

Therefore, the shortest distance between the given lines is

$$d = \left| \frac{(\bar{a}_2 - \bar{a}_1) \times \bar{b}}{|\bar{b}|} \right|$$

$$\therefore d = \left| \frac{\sqrt{293}}{7} \right|$$

$$d = \frac{\sqrt{293}}{7} \text{ units}$$

14. Question

Find the distance between the parallel lines L_1 and L_2 whose vector equations are

$$\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}), \text{ and } \bar{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k}).$$

Answer

Given equations :

$$\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$\bar{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$$

To Find : d

Formula :

1. Cross Product :

If \bar{a} & \bar{b} are two vectors

$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

2. Dot Product :

If \bar{a} & \bar{b} are two vectors

$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$



then,

$$\bar{a} \cdot \bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

3. Shortest distance between two parallel lines :

The shortest distance between the parallel lines $\bar{r} = \bar{a}_1 + \lambda \bar{b}$ and

$\bar{r} = \bar{a}_2 + \lambda \bar{b}$ is given by,

$$d = \left| \frac{|(\bar{a}_2 - \bar{a}_1) \times \bar{b}|}{|\bar{b}|} \right|$$

Answer :

For given lines,

$$\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$\bar{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$$

Here,

$$\bar{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\bar{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$$

$$\bar{b} = \hat{i} - \hat{j} + \hat{k}$$

$$\therefore |\bar{b}| = \sqrt{1^2 + (-1)^2 + 1^2}$$

$$= \sqrt{1 + 1 + 1}$$

$$= \sqrt{3}$$

$$\bar{a}_2 - \bar{a}_1 = (2 - 1)\hat{i} + (-1 - 2)\hat{j} + (-1 - 3)\hat{k}$$

$$\therefore \bar{a}_2 - \bar{a}_1 = \hat{i} - 3\hat{j} - 4\hat{k}$$

$$(\bar{a}_2 - \bar{a}_1) \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -4 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}(-3 - 4) - \hat{j}(1 + 4) + \hat{k}(-1 + 3)$$

$$\therefore (\bar{a}_2 - \bar{a}_1) \times \bar{b} = -7\hat{i} - 5\hat{j} + 2\hat{k}$$

$$\therefore |(\bar{a}_2 - \bar{a}_1) \times \bar{b}| = \sqrt{(-7)^2 + (-5)^2 + 2^2}$$

$$= \sqrt{49 + 25 + 4}$$

$$= \sqrt{78}$$



Therefore, the shortest distance between the given lines is

$$d = \frac{|(\bar{a}_2 - \bar{a}_1) \times \bar{b}|}{|\bar{b}|}$$

$$\therefore d = \frac{|\sqrt{78}|}{|\sqrt{3}|}$$

$$\therefore d = \sqrt{26}$$

$$d = \sqrt{26} \text{ units}$$

15. Question

Find the vector equation of a line passing through the point (2, 3, 2) and parallel to the line

$$\bar{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k}). \text{ Also, find the distance between these lines.}$$

HINT: The given line is

$$L_1 : \bar{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k}).$$

The required line is

$$L_2 : \bar{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(2\hat{i} - 3\hat{j} + 6\hat{k}).$$

Now, find the distance between the parallel lines L_1 and L_2 .

Answer

Given : point A \equiv (2, 3, 2)

$$\text{Equation of line : } \bar{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$$

To Find : i) equation of line

ii) distance d

Formulae :

1. Equation of line :

Equation of line passing through point A (a_1, a_2, a_3) and parallel to vector $\bar{b} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$\bar{r} = \bar{a} + \lambda\bar{b}$$

$$\text{Where, } \bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

2. Cross Product :

If \bar{a} & \bar{b} are two vectors

$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3. Dot Product :

If \bar{a} & \bar{b} are two vectors

$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

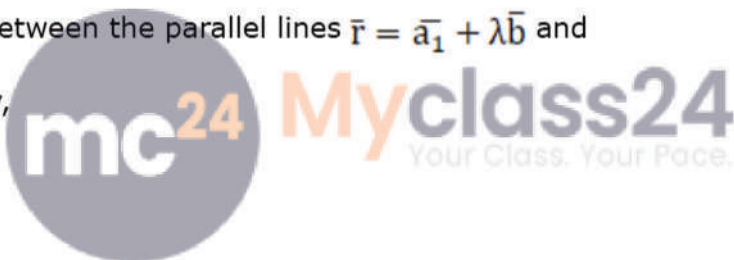
$$\bar{a} \cdot \bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

4. Shortest distance between two parallel lines :

The shortest distance between the parallel lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and

$\bar{r} = \bar{a}_2 + \lambda\bar{b}$ is given by,

$$d = \frac{|(\bar{a}_2 - \bar{a}_1) \times \bar{b}|}{|\bar{b}|}$$



Answer :

As the required line is parallel to the line

$$\bar{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$$

Therefore, the vector parallel to the required line is

$$\bar{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$$

Given point A \equiv (2, 3, 2)

$$\therefore \bar{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$$

Therefore, equation of line passing through A and parallel to \bar{b} is

$$\bar{r} = \bar{a} + \mu\bar{b}$$

$$\therefore \bar{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(2\hat{i} - 3\hat{j} + 6\hat{k})$$

Now, to calculate distance between above line and given line,

$$\bar{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(2\hat{i} - 3\hat{j} + 6\hat{k})$$

$$\bar{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$$

Here,

$$\bar{a}_1 = 2\hat{i} + 3\hat{j} + 2\hat{k}$$

$$\bar{a}_2 = -2\hat{i} + 3\hat{j}$$

$$\bar{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$$

$$\therefore |\bar{b}| = \sqrt{2^2 + (-3)^2 + 6^2}$$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49}$$

$$= 7$$

$$\bar{a}_2 - \bar{a}_1 = (-2 - 2)\hat{i} + (3 - 3)\hat{j} + (0 - 2)\hat{k}$$

$$\therefore \bar{a}_2 - \bar{a}_1 = -4\hat{i} + 0\hat{j} - 2\hat{k}$$

$$(\bar{a}_2 - \bar{a}_1) \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 0 & -2 \\ 2 & -3 & 6 \end{vmatrix}$$

$$= \hat{i}(0 - 6) - \hat{j}(-24 + 4) + \hat{k}(12 - 0)$$

$$\therefore (\bar{a}_2 - \bar{a}_1) \times \bar{b} = -6\hat{i} + 20\hat{j} + 12\hat{k}$$

$$\therefore |(\bar{a}_2 - \bar{a}_1) \times \bar{b}| = \sqrt{(-6)^2 + 20^2 + 12^2}$$

$$= \sqrt{36 + 400 + 144}$$

$$= \sqrt{580}$$

Therefore, the shortest distance between the given lines is

$$d = \frac{|(\bar{a}_2 - \bar{a}_1) \times \bar{b}|}{|\bar{b}|}$$

$$\therefore d = \frac{\sqrt{580}}{7}$$

$$\therefore d = \frac{\sqrt{580}}{7}$$

$$d = \frac{\sqrt{580}}{7} \text{ units}$$

16. Question



Write the vector equation of each of the following lines and hence determine the distance between them :

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6} \text{ and } \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}.$$

HINT: The given lines are

$$L_1 : \vec{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$$

$$L_2 : \vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Now, find the distance between the parallel lines L_1 and L_2 .

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

$$L2 : \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}$$

To Find : i) vector equations of given lines

ii) distance d



Formulae :

1. Equation of line :

Equation of line passing through point A (a_1, a_2, a_3) and having direction ratios (b_1, b_2, b_3) is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

$$\text{Where, } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\text{And } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

2. Cross Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3. Dot Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \cdot \vec{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

4. Shortest distance between two parallel lines :

The shortest distance between the parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and

$\vec{r} = \vec{a}_2 + \lambda\vec{b}$ is given by,

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$$

Answer :

Given Cartesian equations of lines

$$L1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

Line L1 is passing through point (1, 2, -4) and has direction ratios (2, 3, 6)

Therefore, vector equation of line L1 is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

And

$$L2 : \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}$$

Line L2 is passing through point (3, 3, -5) and has direction ratios (4, 6, 12)

Therefore, vector equation of line L2 is

$$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$$

$$\therefore \vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Now, to calculate distance between the lines,

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Here,

$$\bar{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}$$

$$\bar{b}_1 = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\bar{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\bar{b}_2 = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

As $\bar{b}_1 = \bar{b}_2 = \bar{b}$ (say), given lines are parallel to each other.

Therefore,

$$\bar{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\therefore |\bar{b}| = \sqrt{2^2 + 3^2 + 6^2}$$

$$= \sqrt{4 + 9 + 36}$$

$$= \sqrt{49}$$

$$= 7$$

$$\bar{a}_2 - \bar{a}_1 = (3 - 1)\hat{i} + (3 - 2)\hat{j} + (-5 + 4)\hat{k}$$

$$\therefore \bar{a}_2 - \bar{a}_1 = 2\hat{i} + \hat{j} - \hat{k}$$

$$(\bar{a}_2 - \bar{a}_1) \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix}$$

$$= \hat{i}(6 + 3) - \hat{j}(12 + 2) + \hat{k}(6 - 2)$$

$$\therefore (\bar{a}_2 - \bar{a}_1) \times \bar{b} = 9\hat{i} - 14\hat{j} + 4\hat{k}$$

$$\therefore |(\bar{a}_2 - \bar{a}_1) \times \bar{b}| = \sqrt{9^2 + (-14)^2 + 4^2}$$

$$= \sqrt{81 + 196 + 16}$$

$$= \sqrt{293}$$

Therefore, the shortest distance between the given lines is

$$d = \left| \frac{|(\bar{a}_2 - \bar{a}_1) \times \bar{b}|}{|\bar{b}|} \right|$$

$$\therefore d = \left| \frac{\sqrt{293}}{7} \right|$$

$$d = \frac{\sqrt{293}}{7} \text{ units}$$



17. Question

Write the vector equation of the following lines and hence find the shortest distance between them :

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-5}{5}.$$

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$L2 : \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-5}{5}$$

To Find : i) vector equations of given lines

ii) distance d

Formulae :

1. Equation of line :

Equation of line passing through point A (a_1, a_2, a_3) and having direction ratios (b_1, b_2, b_3) is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

Where, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

And $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

2. Cross Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

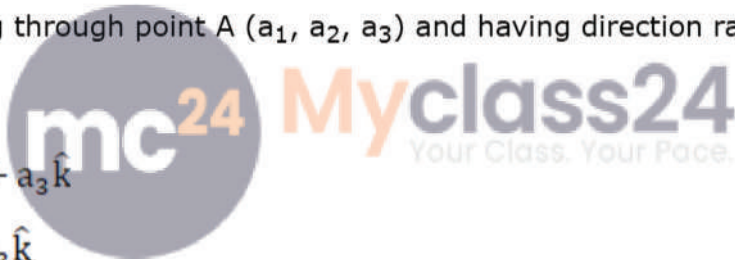
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3. Dot Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$



then,

$$\bar{a} \cdot \bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

4. Shortest distance between two lines :

The shortest distance between the skew lines $\bar{r} = \bar{a}_1 + \lambda \bar{b}_1$ and

$\bar{r} = \bar{a}_2 + \lambda \bar{b}_2$ is given by,

$$d = \left| \frac{(\bar{b}_1 \times \bar{b}_2) \cdot (\bar{a}_2 - \bar{a}_1)}{|\bar{b}_1 \times \bar{b}_2|} \right|$$

Answer :

Given Cartesian equations of lines

$$L1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

Line L1 is passing through point (1, 2, 3) and has direction ratios (2, 3, 4)

Therefore, vector equation of line L1 is

$$\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$$

And

$$L2 : \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-5}{5}$$

Line L2 is passing through point (2, 3, 5) and has direction ratios (3, 4, 5)

Therefore, vector equation of line L2 is

$$\bar{r} = (3\hat{i} + 3\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k})$$

Now, to calculate distance between the lines,

$$\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$$

$$\bar{r} = (3\hat{i} + 3\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k})$$

Here,

$$\bar{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\bar{b}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\bar{a}_2 = 3\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\bar{b}_2 = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

Therefore,

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= \hat{i}(15 - 16) - \hat{j}(10 - 12) + \hat{k}(8 - 9)$$

$$\therefore \vec{b}_1 \times \vec{b}_2 = -\hat{i} + 2\hat{j} - \hat{k}$$

$$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-1)^2 + 2^2 + (-1)^2}$$

$$= \sqrt{1 + 4 + 1}$$

$$= \sqrt{6}$$

$$\vec{a}_2 - \vec{a}_1 = (3 - 1)\hat{i} + (3 - 2)\hat{j} + (5 - 3)\hat{k}$$

$$\therefore \vec{a}_2 - \vec{a}_1 = 2\hat{i} + \hat{j} + 2\hat{k}$$

Now,

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (-\hat{i} + 2\hat{j} - \hat{k}) \cdot (2\hat{i} + \hat{j} + 2\hat{k})$$

$$= ((-1) \times 2) + (2 \times 1) + ((-1) \times 2)$$

$$= -2 + 2 - 2$$

$$= -2$$

Therefore, the shortest distance between the given lines is

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

$$\therefore d = \left| \frac{-2}{\sqrt{6}} \right|$$

$$\therefore d = \frac{2}{\sqrt{3} \cdot \sqrt{2}}$$

$$\therefore d = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\therefore d = \sqrt{\frac{2}{3}}$$

$$d = \sqrt{\frac{2}{3}} \text{ units}$$

18. Question



Find the shortest distance between the lines given below:

$$\frac{x-1}{-1} = \frac{y+2}{1} = \frac{z-3}{-2} \text{ and } \frac{x-1}{2} = \frac{y+1}{2} = \frac{z+1}{-2}.$$

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x-1}{-1} = \frac{y+2}{1} = \frac{z-3}{-2}$$

$$L2 : \frac{x-1}{2} = \frac{y+1}{2} = \frac{z+1}{-2}$$

To Find : distance d

Formulae :

1. Equation of line :

Equation of line passing through point A (a_1, a_2, a_3) and having direction ratios (b_1, b_2, b_3) is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

Where, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

And $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

2. Cross Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3. Dot Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \cdot \vec{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$



4. Shortest distance between two lines :

The shortest distance between the skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and

$\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is given by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Answer :

Given Cartesian equations of lines

$$L1 : \frac{x-1}{-1} = \frac{y+2}{1} = \frac{z-3}{-2}$$

Line L1 is passing through point (1, -2, 3) and has direction ratios (-1, 1, -2)

Therefore, vector equation of line L1 is

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + \hat{j} - 2\hat{k})$$

And

$$L2 : \frac{x-1}{2} = \frac{y+1}{2} = \frac{z+1}{-2}$$

Line L2 is passing through point (1, -1, -1) and has direction ratios (2, 2, -2)

Therefore, vector equation of line L2 is

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + 2\hat{j} - 2\hat{k})$$

Now, to calculate distance between the lines,

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + \hat{j} - 2\hat{k})$$

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + 2\hat{j} - 2\hat{k})$$

Here,

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$$

$$\vec{b}_2 = 2\hat{i} + 2\hat{j} - 2\hat{k}$$

Therefore,

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 2 & 2 & -2 \end{vmatrix}$$

$$= \hat{i}(-2 + 4) - \hat{j}(2 + 4) + \hat{k}(-2 - 2)$$

$$\therefore \vec{b}_1 \times \vec{b}_2 = 2\hat{i} - 6\hat{j} - 4\hat{k}$$

$$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{2^2 + (-6)^2 + (-4)^2}$$

$$= \sqrt{4 + 36 + 16}$$

$$= \sqrt{56}$$

$$\vec{a}_2 - \vec{a}_1 = (1 - 1)\hat{i} + (-1 + 2)\hat{j} + (-1 - 3)\hat{k}$$

$$\therefore \vec{a}_2 - \vec{a}_1 = 0\hat{i} + \hat{j} - 4\hat{k}$$

Now,

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (2\hat{i} - 6\hat{j} - 4\hat{k}) \cdot (0\hat{i} + \hat{j} - 4\hat{k})$$

$$= (2 \times 0) + ((-6) \times 1) + ((-4) \times (-4))$$

$$= 0 - 6 + 16$$

$$= 10$$

Therefore, the shortest distance between the given lines is

$$d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}$$

$$\therefore d = \left| \frac{10}{\sqrt{56}} \right|$$

$$\therefore d = \frac{10}{\sqrt{56}}$$

$$d = \frac{10}{\sqrt{56}} \text{ units}$$

19. Question

Find the shortest distance between the lines given below:

$$\frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2} \text{ and } \frac{x-23}{-6} = \frac{y-10}{-4} = \frac{z-25}{3}$$

HINT: Change the given equations in vector form.

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2}$$

$$L2 : \frac{x-23}{-6} = \frac{y-10}{-4} = \frac{z-23}{3}$$

To Find : distance d

Formulae :

1. Equation of line :

Equation of line passing through point A (a_1, a_2, a_3) and having direction ratios (b_1, b_2, b_3) is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

Where, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

And $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

2. Cross Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



3. Dot Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \cdot \vec{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

4. Shortest distance between two lines :

The shortest distance between the skew lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and

$\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ is given by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Answer :

Given Cartesian equations of lines

$$L1 : \frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2}$$

Line L1 is passing through point (12, 1, 5) and has direction ratios (-9, 4, 2)

Therefore, vector equation of line L1 is

$$\vec{r} = (12\hat{i} + \hat{j} + 5\hat{k}) + \lambda(-9\hat{i} + 4\hat{j} + 2\hat{k})$$

And

$$L2 : \frac{x-23}{-6} = \frac{y-10}{-4} = \frac{z-23}{3}$$

Line L2 is passing through point (23, 10, 23) and has direction ratios (-6, -4, 3)

Therefore, vector equation of line L2 is

$$\vec{r} = (23\hat{i} + 10\hat{j} + 23\hat{k}) + \mu(-6\hat{i} - 4\hat{j} + 3\hat{k})$$

Now, to calculate distance between the lines,

$$\vec{r} = (12\hat{i} + \hat{j} + 5\hat{k}) + \lambda(-9\hat{i} + 4\hat{j} + 2\hat{k})$$

$$\vec{r} = (23\hat{i} + 10\hat{j} + 23\hat{k}) + \mu(-6\hat{i} - 4\hat{j} + 3\hat{k})$$

Here,

$$\vec{a}_1 = 12\hat{i} + \hat{j} + 5\hat{k}$$

$$\vec{b}_1 = -9\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\vec{a}_2 = 23\hat{i} + 10\hat{j} + 23\hat{k}$$

$$\vec{b}_2 = -6\hat{i} - 4\hat{j} + 3\hat{k}$$

Therefore,

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -9 & 4 & 2 \\ -6 & -4 & 3 \end{vmatrix}$$

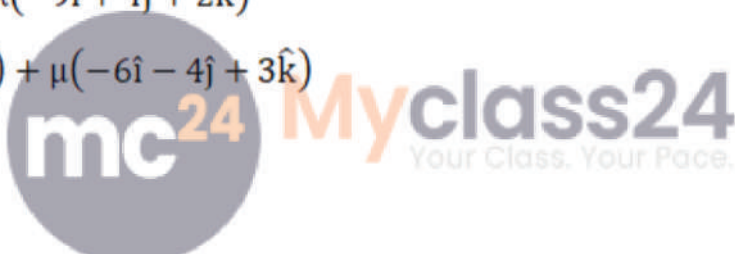
$$= \hat{i}(12 + 8) - \hat{j}(-27 + 12) + \hat{k}(36 + 24)$$

$$\therefore \vec{b}_1 \times \vec{b}_2 = 20\hat{i} + 15\hat{j} + 60\hat{k}$$

$$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{20^2 + 15^2 + 60^2}$$

$$= \sqrt{400 + 225 + 3600}$$

$$= \sqrt{4225}$$



$$= 65$$

$$\vec{a}_2 - \vec{a}_1 = (23 - 12)\hat{i} + (10 - 1)\hat{j} + (23 - 5)\hat{k}$$

$$\therefore \vec{a}_2 - \vec{a}_1 = 11\hat{i} + 9\hat{j} + 18\hat{k}$$

Now,

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (20\hat{i} + 15\hat{j} + 60\hat{k}) \cdot (11\hat{i} + 9\hat{j} + 18\hat{k})$$

$$= (20 \times 11) + (15 \times 9) + (60 \times 18)$$

$$= 220 + 135 + 1080$$

$$= 1435$$

Therefore, the shortest distance between the given lines is

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

$$\therefore d = \left| \frac{1435}{65} \right|$$

$$\therefore d = \frac{287}{13}$$

$$d = \frac{287}{13} \text{ units}$$



Exercise 27E

1. Question

Find the length and the equations of the line of shortest distance between the lines given by:

$$\frac{x-3}{3} = \frac{y-8}{-1} = z-3 \text{ and } \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}.$$

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

$$L2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

Formulae :

1. Condition for perpendicularity :

If line L1 has direction ratios (a_1, a_2, a_3) and that of line L2 are (b_1, b_2, b_3) then lines L1 and L2 will be perpendicular to each other if

$$(a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3) = 0$$

2. Distance formula :

Distance between two points $A \equiv (a_1, a_2, a_3)$ and $B \equiv (b_1, b_2, b_3)$ is given by,

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

3. Equation of line :

Equation of line passing through points $A \equiv (x_1, y_1, z_1)$ and $B \equiv (x_2, y_2, z_2)$ is given by,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Answer :

Given equations of lines

$$L1 : \frac{x - 3}{3} = \frac{y - 8}{-1} = \frac{z - 3}{1}$$

$$L2 : \frac{x + 3}{-3} = \frac{y + 7}{2} = \frac{z - 6}{4}$$

Direction ratios of L1 and L2 are $(3, -1, 1)$ and $(-3, 2, 4)$ respectively.

Let, general point on line L1 is $P \equiv (x_1, y_1, z_1)$

$$x_1 = 3s + 3, y_1 = -s + 8, z_1 = s + 3$$

and let, general point on line L2 is $Q \equiv (x_2, y_2, z_2)$

$$x_2 = -3t - 3, y_2 = 2t - 7, z_2 = 4t + 6$$

$$\begin{aligned} \therefore \overline{PQ} &= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \\ &= (-3t - 3 - 3s - 3)\hat{i} + (2t - 7 + s - 8)\hat{j} + (4t + 6 - s - 3)\hat{k} \end{aligned}$$

$$\therefore \overline{PQ} = (-3t - 3s - 6)\hat{i} + (2t + s - 15)\hat{j} + (4t - s + 3)\hat{k}$$

Direction ratios of \overline{PQ} are $(-3t - 3s - 6), (2t + s - 15), (4t - s + 3)$

PQ will be the shortest distance if it perpendicular to both the given lines

Therefore, by the condition of perpendicularity,

$$3(-3t - 3s - 6) - 1(2t + s - 15) + 1(4t - s + 3) = 0 \text{ and}$$

$$-3(-3t - 3s - 6) + 2(2t + s - 15) + 4(4t - s + 3) = 0$$

$$\Rightarrow -9t - 9s - 18 - 2t - s + 15 + 4t - s + 3 = 0 \text{ and}$$

$$9t + 9s + 18 + 4t + 2s - 30 + 16t - 4s + 12 = 0$$

$$\Rightarrow -7t - 11s = 0 \text{ and}$$

$$29t + 7s = 0$$

Solving above two equations, we get,

$$t = 0 \text{ and } s = 0$$

therefore,

$$P \equiv (3, 8, 3) \text{ and } Q \equiv (-3, -7, 6)$$

Now, distance between points P and Q is

$$\begin{aligned} d &= \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} \\ &= \sqrt{(6)^2 + (15)^2 + (-3)^2} \\ &= \sqrt{36 + 225 + 9} \\ &= \sqrt{270} \\ &= 3\sqrt{30} \end{aligned}$$

Therefore, the shortest distance between two given lines is

$$d = 3\sqrt{30} \text{ units}$$

Now, equation of line passing through points P and Q is,

$$\begin{aligned} \frac{x-x_1}{x_1-x_2} &= \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2} \\ \therefore \frac{x-3}{3+3} &= \frac{y-8}{8+7} = \frac{z-3}{3-6} \\ \therefore \frac{x-3}{6} &= \frac{y-8}{15} = \frac{z-3}{-3} \\ \therefore \frac{x-3}{2} &= \frac{y-8}{5} = \frac{z-3}{-1} \end{aligned}$$

Therefore, equation of line of shortest distance between two given lines is

$$\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

2. Question

Find the length and the equations of the line of shortest distance between the lines given by:

$$\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1} \text{ and } \frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}$$

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}$$

$$L2 : \frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}$$

Formulae :

1. Condition for perpendicularity :

If line L1 has direction ratios (a_1, a_2, a_3) and that of line L2 are (b_1, b_2, b_3) then lines L1 and L2 will be perpendicular to each other if

$$(a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3) = 0$$

2. Distance formula :

Distance between two points $A \equiv (a_1, a_2, a_3)$ and $B \equiv (b_1, b_2, b_3)$ is given by,

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

3. Equation of line :

Equation of line passing through points $A \equiv (x_1, y_1, z_1)$ and $B \equiv (x_2, y_2, z_2)$ is given by,

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Answer :

Given equations of lines

$$L1 : \frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}$$

$$L2 : \frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}$$

Direction ratios of L1 and L2 are $(-1, 2, 1)$ and $(1, 3, 2)$ respectively.

Let, general point on line L1 is $P \equiv (x_1, y_1, z_1)$

$$x_1 = -s+3, y_1 = 2s+4, z_1 = s-2$$

and let, general point on line L2 is $Q \equiv (x_2, y_2, z_2)$

$$x_2 = t+1, y_2 = 3t-7, z_2 = 2t-2$$

$$\therefore \overline{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$= (t+1+s-3)\hat{i} + (3t-7-2s-4)\hat{j} + (2t-2-s+2)\hat{k}$$

$$\therefore \overline{PQ} = (t + s - 2)\hat{i} + (3t - 2s - 11)\hat{j} + (2t - s)\hat{k}$$

Direction ratios of \overline{PQ} are $((t + s - 2), (3t - 2s - 11), (2t - s))$

PQ will be the shortest distance if it perpendicular to both the given lines

Therefore, by the condition of perpendicularity,

$$-1(t + s - 2) + 2(3t - 2s - 11) + 1(2t - s) = 0 \text{ and}$$

$$1(t + s - 2) + 3(3t - 2s - 11) + 2(2t - s) = 0$$

$$\Rightarrow -t - s + 2 + 6t - 4s - 22 + 2t - s = 0 \text{ and}$$

$$t + s - 2 + 9t - 6s - 33 + 4t - 2s = 0$$

$$\Rightarrow 7t - 6s = 20 \text{ and}$$

$$14t - 7s = 35$$

Solving above two equations, we get,

$$t = 2 \text{ and } s = -1$$

therefore,

$$P \equiv (4, 2, -3) \text{ and } Q \equiv (3, -1, 2)$$

Now, distance between points P and Q is

$$d = \sqrt{(4 - 3)^2 + (2 + 1)^2 + (-3 - 2)^2}$$

$$= \sqrt{(1)^2 + (3)^2 + (-5)^2}$$

$$= \sqrt{1 + 9 + 25}$$

$$= \sqrt{35}$$

Therefore, the shortest distance between two given lines is

$$d = \sqrt{35} \text{ units}$$

Now, equation of line passing through points P and Q is,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\therefore \frac{x - 4}{4 - 3} = \frac{y - 2}{2 + 1} = \frac{z + 3}{-3 - 2}$$

$$\therefore \frac{x - 4}{1} = \frac{y - 2}{3} = \frac{z + 3}{-5}$$

$$\therefore \frac{x - 4}{-1} = \frac{y - 2}{-3} = \frac{z + 3}{5}$$



Therefore, equation of line of shortest distance between two given lines is

$$\frac{x-4}{-1} = \frac{y-2}{-3} = \frac{z+3}{5}$$

3. Question

Find the length and the equations of the line of shortest distance between the lines given by:

$$\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3} \text{ and } \frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$$

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$$

$$L2 : \frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$$

Formulae :

1. Condition for perpendicularity :

If line L1 has direction ratios (a_1, a_2, a_3) and that of line L2 are (b_1, b_2, b_3) then lines L1 and L2 will be perpendicular to each other if

$$(a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3) = 0$$

2. Distance formula :

Distance between two points $A \equiv (a_1, a_2, a_3)$ and $B \equiv (b_1, b_2, b_3)$ is given by,

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

3. Equation of line :

Equation of line passing through points $A \equiv (x_1, y_1, z_1)$ and $B \equiv (x_2, y_2, z_2)$ is given by,

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2}$$

Answer :

Given equations of lines

$$L1 : \frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$$

$$L2 : \frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$$

Direction ratios of L1 and L2 are $(2, 1, -3)$ and $(2, -7, 5)$ respectively.

Let, general point on line L1 is $P \equiv (x_1, y_1, z_1)$

$$x_1 = 2s-1, y_1 = s+1, z_1 = -3s+9$$

and let, general point on line L2 is $Q \equiv (x_2, y_2, z_2)$

$$x_2 = 2t+3, y_2 = -7t-15, z_2 = 5t+9$$

$$\begin{aligned} \therefore \overline{PQ} &= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \\ &= (5t+9-2s+1)\hat{i} + (-7t-15-s-1)\hat{j} + (5t+9+3s-9)\hat{k} \end{aligned}$$

$$\therefore \overline{PQ} = (5t-2s+10)\hat{i} + (-7t-s-16)\hat{j} + (5t+3s)\hat{k}$$

Direction ratios of \overline{PQ} are $((5t-2s+10), (-7t-s-16), (5t+3s))$

PQ will be the shortest distance if it perpendicular to both the given lines

Therefore, by the condition of perpendicularity,

$$2(5t-2s+10) + 1(-7t-s-16) - 3(5t+3s) = 0 \text{ and}$$

$$2(5t-2s+10) - 7(-7t-s-16) + 5(5t+3s) = 0$$

$$\Rightarrow 10t - 4s + 20 - 7t - s - 16 - 15t - 9s = 0 \text{ and}$$

$$10t - 4s + 20 + 49t + 7s + 112 + 25t + 15s = 0$$

$$\Rightarrow -12t - 14s = -4 \text{ and}$$

$$84t + 18s = -132$$

Solving above two equations, we get,

$$t = -2 \text{ and } s = 2$$

therefore,

$$P \equiv (3, 3, 3) \text{ and } Q \equiv (-1, -1, -1)$$

Now, distance between points P and Q is

$$d = \sqrt{(3+1)^2 + (3+1)^2 + (3+1)^2}$$

$$= \sqrt{(4)^2 + (4)^2 + (4)^2}$$

$$= \sqrt{16 + 16 + 16}$$

$$= \sqrt{48}$$

$$= 4\sqrt{3}$$

Therefore, the shortest distance between two given lines is

$$d = 4\sqrt{3} \text{ units}$$



Now, equation of line passing through points P and Q is,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\therefore \frac{x - 3}{3 + 1} = \frac{y - 3}{3 + 1} = \frac{z - 3}{3 + 1}$$

$$\therefore \frac{x - 3}{4} = \frac{y - 3}{4} = \frac{z - 3}{4}$$

$$\therefore x - 3 = y - 3 = z - 3$$

$$\Rightarrow x = y = z$$

Therefore, equation of line of shortest distance between two given lines is

$$x = y = z$$

4. Question

Find the length and the equations of the line of shortest distance between the lines given by:

$$\frac{x - 6}{3} = \frac{y - 7}{-1} = \frac{z - 4}{1} \text{ and } \frac{x}{-3} = \frac{y + 9}{2} = \frac{z - 2}{4}.$$

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x - 6}{3} = \frac{y - 7}{-1} = \frac{z - 4}{1}$$

$$L2 : \frac{x}{-3} = \frac{y + 9}{2} = \frac{z - 2}{4}$$

Formulae :

1. Condition for perpendicularity :

If line L1 has direction ratios (a_1, a_2, a_3) and that of line L2 are (b_1, b_2, b_3) then lines L1 and L2 will be perpendicular to each other if

$$(a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3) = 0$$

2. Distance formula :

Distance between two points $A \equiv (a_1, a_2, a_3)$ and $B \equiv (b_1, b_2, b_3)$ is given by,

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

3. Equation of line :

Equation of line passing through points $A \equiv (x_1, y_1, z_1)$ and $B \equiv (x_2, y_2, z_2)$ is given by,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

Answer :

Given equations of lines

$$L1 : \frac{x - 6}{3} = \frac{y - 7}{-1} = \frac{z - 4}{1}$$

$$L2 : \frac{x}{-3} = \frac{y + 9}{2} = \frac{z - 2}{4}$$

Direction ratios of L1 and L2 are (3, -1, 1) and (-3, 2, 4) respectively.

Let, general point on line L1 is $P \equiv (x_1, y_1, z_1)$

$$x_1 = 3s + 6, y_1 = -s + 7, z_1 = s + 4$$

and let, general point on line L2 is $Q \equiv (x_2, y_2, z_2)$

$$x_2 = -3t, y_2 = 2t - 9, z_2 = 4t + 2$$

$$\therefore \overline{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$= (-3t - 3s - 6)\hat{i} + (2t - 9 + s - 7)\hat{j} + (4t + 2 - s - 4)\hat{k}$$

$$\therefore \overline{PQ} = (-3t - 3s - 6)\hat{i} + (2t + s - 16)\hat{j} + (4t - s - 2)\hat{k}$$

Direction ratios of \overline{PQ} are $((-3t - 3s - 6), (2t + s - 16), (4t - s - 2))$

PQ will be the shortest distance if it perpendicular to both the given lines

Therefore, by the condition of perpendicularity,

$$3(-3t - 3s - 6) - 1(2t + s - 16) + 1(4t - s - 2) = 0 \text{ and}$$

$$-3(-3t - 3s - 6) + 2(2t + s - 16) + 4(4t - s - 2) = 0$$

$$\Rightarrow -9t - 9s - 18 - 2t - s + 16 + 4t - s - 2 = 0 \text{ and}$$

$$9t + 9s + 18 + 4t + 2s - 32 + 16t - 4s - 8 = 0$$

$$\Rightarrow -7t - 11s = 4 \text{ and}$$

$$29t + 7s = -22$$

Solving above two equations, we get,

$$t = 1 \text{ and } s = -1$$

therefore,

$$P \equiv (3, 8, 3) \text{ and } Q \equiv (-3, -7, 6)$$

Now, distance between points P and Q is

$$\begin{aligned}
 d &= \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} \\
 &= \sqrt{(6)^2 + (15)^2 + (-3)^2} \\
 &= \sqrt{36 + 225 + 9} \\
 &= \sqrt{270} \\
 &= 3\sqrt{30}
 \end{aligned}$$

Therefore, the shortest distance between two given lines is

$$d = 3\sqrt{30} \text{ units}$$

Now, equation of line passing through points P and Q is,

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2}$$

$$\therefore \frac{x-3}{3+3} = \frac{y-8}{8+7} = \frac{z-3}{3-6}$$

$$\therefore \frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$$

$$\therefore \frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$



Therefore, equation of line of shortest distance between two given lines is

$$\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

5. Question

Show that the lines $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ and $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$ intersect and find their point of intersection.

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$$

$$L2 : \frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$$

To Find : distance d

Formulae :

1. Equation of line :

Equation of line passing through point A (a_1, a_2, a_3) and having direction ratios (b_1, b_2, b_3) is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

Where, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

And $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

2. Cross Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3. Dot Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \cdot \vec{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

4. Shortest distance between two lines :

The shortest distance between the skew lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and

$\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ is given by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Answer :

Given Cartesian equations of lines

$$L1 : \frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$$

Line L1 is passing through point (0, 2, -3) and has direction ratios (1, 2, 3)

Therefore, vector equation of line L1 is



$$\bar{r} = (0\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$$

And

$$L2 : \frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$$

Line L2 is passing through point (2, 6, 3) and has direction ratios (2, 3, 4)

Therefore, vector equation of line L2 is

$$\bar{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$$

Now, to calculate distance between the lines,

$$\bar{r} = (0\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\bar{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$$

Here,

$$\bar{a}_1 = 0\hat{i} + 2\hat{j} - 3\hat{k}$$

$$\bar{b}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\bar{a}_2 = 2\hat{i} + 6\hat{j} + 3\hat{k}$$

$$\bar{b}_2 = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

Therefore,

$$\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= \hat{i}(8 - 9) - \hat{j}(4 - 6) + \hat{k}(3 - 4)$$

$$\therefore \bar{b}_1 \times \bar{b}_2 = -\hat{i} + 2\hat{j} - \hat{k}$$

$$\therefore |\bar{b}_1 \times \bar{b}_2| = \sqrt{(-1)^2 + 2^2 + (-1)^2}$$

$$= \sqrt{1 + 4 + 1}$$

$$= \sqrt{6}$$

$$\bar{a}_2 - \bar{a}_1 = (2 - 0)\hat{i} + (6 - 2)\hat{j} + (3 - (-3))\hat{k}$$

$$\therefore \bar{a}_2 - \bar{a}_1 = 2\hat{i} + 4\hat{j} + 6\hat{k}$$

Now,

$$(\bar{b}_1 \times \bar{b}_2) \cdot (\bar{a}_2 - \bar{a}_1) = (-\hat{i} + 2\hat{j} - \hat{k}) \cdot (2\hat{i} + 4\hat{j} + 6\hat{k})$$

$$= ((-1) \times 2) + (2 \times 4) + ((-1) \times 6)$$



$$= -2 + 8 - 6$$

$$= 0$$

Therefore, the shortest distance between the given lines is

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

$$\therefore d = \left| \frac{0}{\sqrt{14}} \right|$$

$$\therefore d = 0 \text{ units}$$

As $d = 0$

Hence, given lines intersect each other.

Now, general point on L1 is

$$x_1 = \lambda, y_1 = 2\lambda + 2, z_1 = 3\lambda - 3$$

let, $P(x_1, y_1, z_1)$ be point of intersection of two given lines.

Therefore, point P satisfies equation of line L2.

$$\therefore \frac{\lambda - 2}{2} = \frac{2\lambda + 2 - 6}{3} = \frac{3\lambda - 3 - 3}{4}$$

$$\therefore \frac{\lambda - 2}{2} = \frac{2\lambda - 4}{3}$$

$$\Rightarrow 3\lambda - 6 = 4\lambda - 8$$

$$\Rightarrow \lambda = 2$$

Therefore, $x_1 = 2, y_1 = 2(2) + 2, z_1 = 3(2) - 3$

$$\Rightarrow x_1 = 2, y_1 = 6, z_1 = 3$$

Hence point of intersection of given lines is $(2, 6, 3)$.

6. Question

Show that the lines $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$ and $\frac{x-2}{2} = \frac{y-1}{3} = \frac{z+1}{-2}$ do not intersect each other.

Answer

Given : Cartesian equations of lines

$$L1 : \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$$

$$L2 : \frac{x-2}{2} = \frac{y-1}{3} = \frac{z+1}{-2}$$

To Find : distance d

Formulae :

1. Equation of line :

Equation of line passing through point A (a_1, a_2, a_3) and having direction ratios (b_1, b_2, b_3) is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

Where, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

And $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

2. Cross Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



3. Dot Product :

If \vec{a} & \vec{b} are two vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

then,

$$\vec{a} \cdot \vec{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

4. Shortest distance between two lines :

The shortest distance between the skew lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and

$\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ is given by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Answer :

Given Cartesian equations of lines

$$L1 : \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$$

Line L1 is passing through point (1, -1, 1) and has direction ratios (3, 2, 5)

Therefore, vector equation of line L1 is

$$\vec{r} = (\hat{i} - \hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 5\hat{k})$$

And

$$L2 : \frac{x-2}{2} = \frac{y-1}{3} = \frac{z+1}{-2}$$

Line L2 is passing through point (2, 1, -1) and has direction ratios (2, 3, -2)

Therefore, vector equation of line L2 is

$$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(2\hat{i} + 3\hat{j} - 2\hat{k})$$

Now, to calculate distance between the lines,

$$\vec{r} = (\hat{i} - \hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 5\hat{k})$$

$$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(2\hat{i} + 3\hat{j} - 2\hat{k})$$

Here,

$$\vec{a}_1 = \hat{i} - \hat{j} + \hat{k}$$

$$\vec{b}_1 = 3\hat{i} + 2\hat{j} + 5\hat{k}$$

$$\vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k}$$

$$\vec{b}_2 = 2\hat{i} + 3\hat{j} - 2\hat{k}$$

Therefore,

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 5 \\ 2 & 3 & -2 \end{vmatrix}$$

$$= \hat{i}(-4 - 15) - \hat{j}(-6 - 10) + \hat{k}(9 - 4)$$

$$\therefore \vec{b}_1 \times \vec{b}_2 = -19\hat{i} + 16\hat{j} + 5\hat{k}$$

$$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-19)^2 + 16^2 + 5^2}$$

$$= \sqrt{361 + 256 + 25}$$

$$= \sqrt{642}$$



$$\vec{a}_2 - \vec{a}_1 = (2 - 1)\hat{i} + (1 + 1)\hat{j} + (-1 - 1)\hat{k}$$

$$\therefore \vec{a}_2 - \vec{a}_1 = \hat{i} + 2\hat{j} - 2\hat{k}$$

Now,

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (-19\hat{i} + 16\hat{j} + 5\hat{k}) \cdot (\hat{i} + 2\hat{j} - 2\hat{k})$$

$$= ((-19) \times 1) + (16 \times 2) + (5 \times (-2))$$

$$= -19 + 32 - 10$$

$$= 3$$

Therefore, the shortest distance between the given lines is

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

$$\therefore d = \left| \frac{3}{\sqrt{642}} \right|$$

$$\therefore d = \frac{3}{\sqrt{642}} \text{ units}$$

As $d \neq 0$

Hence, given lines do not intersect each other.



Exercise 27F

1. Question

If a line has direction ratios 2, -1, -2 then what are its direction cosines?

Answer

Given : A line has direction ratios 2, -1, -2

To find : Direction cosines of the line

Formula used : If (l, m, n) are the direction ratios of a given line then direction cosines are given by

$$\frac{l}{\sqrt{l^2 + m^2 + n^2}}, \frac{m}{\sqrt{l^2 + m^2 + n^2}}, \frac{n}{\sqrt{l^2 + m^2 + n^2}}$$

Here $l = 2$, $m = -1$, $n = -2$

Direction cosines of the line with direction ratios 2, -1, -2 is

$$\frac{2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-1}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$= \frac{2}{\sqrt{4+1+4}}, \frac{-1}{\sqrt{4+1+4}}, \frac{-2}{\sqrt{4+1+4}} = \frac{2}{\sqrt{9}}, \frac{-1}{\sqrt{9}}, \frac{-2}{\sqrt{9}}$$

$$= \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$$

Direction cosines of the line with direction ratios 2, -1, -2 is $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$

2. Question

Find the direction cosines of the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.

Answer

Given : A line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.

To find : Direction cosines of the line

Formula used : If a line is given by $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ then direction cosines are given by $\frac{l}{\sqrt{l^2+m^2+n^2}}$, $\frac{m}{\sqrt{l^2+m^2+n^2}}$, $\frac{n}{\sqrt{l^2+m^2+n^2}}$

The line is $\frac{x-4}{-2} = \frac{y-0}{6} = \frac{z-1}{-3}$

Here $l = -2$, $m = 6$, $n = -3$

Direction cosines of the line $\frac{x-4}{-2} = \frac{y-0}{6} = \frac{z-1}{-3}$ is

$$\frac{-2}{\sqrt{(-2)^2+(6)^2+(-3)^2}}, \frac{6}{\sqrt{(-2)^2+(6)^2+(-3)^2}}, \frac{-3}{\sqrt{(-2)^2+(6)^2+(-3)^2}}$$
$$= \frac{-2}{\sqrt{4+36+9}}, \frac{6}{\sqrt{4+36+9}}, \frac{-3}{\sqrt{4+36+9}} = \frac{-2}{\sqrt{49}}, \frac{6}{\sqrt{49}}, \frac{-3}{\sqrt{49}}$$
$$= \frac{-2}{7}, \frac{6}{7}, \frac{-3}{7}$$

Direction cosines of the line $\frac{x-4}{-2} = \frac{y-0}{6} = \frac{z-1}{-3}$ is $\frac{-2}{7}, \frac{6}{7}, \frac{-3}{7}$

3. Question

If the equations of a line are $\frac{3-x}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$, find the direction cosines of a line parallel to the given line.

Answer

Given : A line $\frac{3-x}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$,

To find : Direction cosines of the line parallel to $\frac{3-x}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$,

Formula used : If a line is given by $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ then direction cosines are given by $\frac{l}{\sqrt{l^2+m^2+n^2}}$, $\frac{m}{\sqrt{l^2+m^2+n^2}}$, $\frac{n}{\sqrt{l^2+m^2+n^2}}$

The line is $\frac{x-3}{3} = \frac{y+2}{-2} = \frac{z+2}{6}$

Parallel lines have same direction ratios and direction cosines

Here $l = 3$, $m = -2$, $n = 6$

Direction cosines of the line $\frac{x-3}{3} = \frac{y+2}{-2} = \frac{z+2}{6}$ is

$$\frac{3}{\sqrt{(3)^2+(-2)^2+(6)^2}}, \frac{-2}{\sqrt{(3)^2+(-2)^2+(6)^2}}, \frac{6}{\sqrt{(3)^2+(-2)^2+(6)^2}}$$

$$= \frac{3}{\sqrt{9+4+36}}, \frac{-2}{\sqrt{9+4+36}}, \frac{6}{\sqrt{9+4+36}} = \frac{3}{\sqrt{49}}, \frac{-2}{\sqrt{49}}, \frac{6}{\sqrt{49}}$$

$$= \frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$$

Direction cosines of the line parallel to the line $\frac{x-3}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$ is

$$\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$$

4. Question

Write the equations of a line parallel to the line $\frac{x-2}{-3} = \frac{y+3}{2} = \frac{z+5}{6}$ and passing through the point (1, -2, 3).

Answer

Given : A line $\frac{x-2}{-3} = \frac{y+3}{2} = \frac{z+5}{6}$

To find : equations of a line parallel to the line $\frac{x-2}{-3} = \frac{y+3}{2} = \frac{z+5}{6}$ and passing through the point (1, -2, 3).

Formula used : If a line is given by $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ then equation of parallel

line passing through the point (p,q,r) is given by $\frac{x-p}{l} = \frac{y-q}{m} = \frac{z-r}{n}$

Here $l = -3$, $m = 2$, $n = 6$ and $p = 1$, $q = -2$, $r = 3$

The line parallel to the line $\frac{x-2}{-3} = \frac{y+3}{2} = \frac{z+5}{6}$ and passing through the point (1,-2,3)

is given by