# EXERCISE 28.5

**Q1.i**

**Solution:**

We know that,

Shortest distance between lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given by

$$S.D. = \left| \frac{(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| \quad \text{........ (1)}$$

Given:

The equation of lines is,

$$\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = (-3\hat{i} - 7\hat{j} + 6\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$$

Where,

$$\vec{a_1} = (3\hat{i} + 8\hat{j} + 3\hat{k}),\ \vec{b_1} = 3\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a_2} = (-3\hat{i} - 7\hat{j} + 6\hat{k}),\ \vec{b_2} = (-3\hat{i} + 2\hat{j} + 4\hat{k})$$

So now,

$$\vec{a_2} - \vec{a_1} = (-3\hat{i} - 7\hat{j} + 6\hat{k}) - (3\hat{i} + 8\hat{j} + 3\hat{k})$$

$$= -3\hat{i} - 7\hat{j} + 6\hat{k} - 3\hat{i} - 8\hat{j} - 3\hat{k}$$

$$(\vec{a_2} - \vec{a_1}) = -6\hat{i} - 15\hat{j} + 3\hat{k} \quad \text{...... (2)}$$

Let us solve for b, we get

$$(\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

$$= \hat{i}(-4 - 2) - \hat{j}(12 + 3) + \hat{k}(6 - 3)$$

$$= (-6\hat{i} - 15\hat{j} + 3\hat{k}) \quad \text{......... (3)}$$

By solving (2) and (3), we get

$$(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2}) = (-6\hat{i} - 15\hat{j} + 3\hat{k}).(-6\hat{i} - 15\hat{j} + 3\hat{k})$$

$$= (-6)(6) + (-15)(-15) + (3)(3)$$

$$= 36 + 225 + 9$$

$$= 270$$

$$\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{(-6)^2 + (-15)^2 + (3)^2}$$
$$= \sqrt{36 + 225 + 9}$$
$$= \sqrt{270}$$

Now let us substitute the above obtained values in equation (1), to get the shortest distance between given lines,

$$\text{S.D.} = \frac{270}{\sqrt{270}}$$
$$= \sqrt{270}$$
$$= 3\sqrt{30}$$

Hence, the shortest distance is $3\sqrt{30}$ units.

**ii.**

**Solution:**

Given:

Equation of lines is,

$$\vec{r} = \left(3\hat{i} + 5\hat{j} + 7\hat{k}\right) + \lambda\left(\hat{i} - 2\hat{j} + 7\hat{k}\right) \text{ and}$$
$$\vec{r} = \left(-\hat{i} - \hat{j} - \hat{k}\right) + \mu\left(7\hat{i} - 6\hat{j} + \hat{k}\right)$$

We know that,

Shortest distance between lines $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\overrightarrow{b_2}$ is given by

$$\text{S.D.} = \left|\frac{\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right).\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right)}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|}\right| \quad \text{........ (1)}$$

Where,

$$\overrightarrow{a_1} = \left(3\hat{i} + 5\hat{j} + 7\hat{k}\right),\ \overrightarrow{b_1} = \left(\hat{i} - 2\hat{j} + 7\hat{k}\right)$$
$$\overrightarrow{a_2} = \left(-\hat{i} - \hat{j} - \hat{k}\right),\ \overrightarrow{b_2} = \left(7\hat{i} - 6\hat{j} + \hat{k}\right)$$

So now,

$$\overrightarrow{a_2} - \overrightarrow{a_1} = \left(-\hat{i} - \hat{j} - \hat{k}\right) - \left(3\hat{i} + 5\hat{j} + 7\hat{k}\right)$$
$$= -\hat{i} - \hat{j} - \hat{k} - 3\hat{i} - 5\hat{j} - 7\hat{k}$$
$$= -4\hat{i} - 6\hat{j} - 8\hat{k} = -2\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) \quad \text{....... (2)}$$

Let us solve for b, we get

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 7 \\ 7 & -6 & 1 \end{vmatrix}$$

$$= \hat{i}(-2+42) - \hat{j}(1-49) + \hat{k}(-6+14)$$

$$= 40\hat{i} + 48\hat{j} + 8\hat{k}$$

$$= 8(5\hat{i} + 6\hat{j} + \hat{k}) \quad \ldots\ldots (3)$$

By solving (2) and (3), we get

$$(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2}) = \{-2(2\hat{i}+3\hat{j}+4\hat{k})\}\{8(5\hat{i}+6\hat{j}+\hat{k})\}$$

$$= -16[(2)(5)+(3)(6)+(4)(1)]$$

$$= -16[10+18+4]$$

$$= -16\times 32$$

$$= -512$$

$$|\vec{b_1}\times\vec{b_2}| = 8\sqrt{(5)^2+(6)^2+(1)^2}$$

$$= 8\sqrt{25+36+1}$$

$$= 8\sqrt{62}$$

Now let us substitute the above obtained values in equation (1), to get the shortest distance between given lines,

$$S.D. = \left|\frac{-512}{8\sqrt{62}}\right|$$

$$= \frac{512}{\sqrt{3968}}$$

Hence, the shortest distance is 512/√3968 units.

**iii.**

**Solution:**

Given:

Equation of lines is,

$$\vec{r} = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(2\hat{i}+3\hat{j}+4\hat{k}) \text{ and}$$

$$\vec{r} = (2\hat{i}+4\hat{j}+5\hat{k}) + \mu(3\hat{i}+4\hat{j}+5\hat{k})$$

We know that,

Shortest distance between lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by

$$S.D. = \left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| \quad \ldots\ldots\ldots (1)$$

Where,

$\vec{a_1} = (\hat{i} + 2\hat{j} + 3\hat{k})$, $\vec{b_1} = (2\hat{i} + 3\hat{j} + 4\hat{k})$

$\vec{a_2} = (2\hat{i} + 4\hat{j} + 5\hat{k})$, $\vec{b_2} = (3\hat{i} + 4\hat{j} + 5\hat{k})$

So now,

$$(\vec{a_2} - \vec{a_1}) = (2\hat{i} + 4\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= 2\hat{i} + 4\hat{j} + 5\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k}$$
$$(\vec{a_2} - \vec{a_1}) = \hat{i} + 2\hat{j} + 2\hat{k} \quad \text{........ (2)}$$

Let us solve for b, we get

$$(\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$
$$= \hat{i}(15 - 16) - \hat{j}(10 - 12) + \hat{k}(8 - 9)$$
$$(\vec{b_1} \times \vec{b_2}) = -\hat{i} + 2\hat{j} - \hat{k} \quad \text{......... (3)}$$

By solving (2) and (3), we get

$$(\vec{a_2} - \vec{a_1})(\vec{b_1} \times \vec{b_2}) = (\hat{i} + 2\hat{j} + 2\hat{k})(-\hat{i} + 2\hat{j} - \hat{k})$$
$$= (1)(-1) + (2)(2) + (2)(-1)$$
$$= -1 + 4 - 2$$
$$= 1$$
$$|\vec{b_1} \times \vec{b_2}| = \sqrt{(-1)^2 + (2)^2 + (-1)^2}$$
$$= \sqrt{1 + 4 + 1}$$
$$= \sqrt{6}$$

Now let us substitute the above obtained values in equation (1), to get the shortest distance between given lines,

$$\text{S.D.} = \left|\frac{1}{\sqrt{6}}\right|$$

Hence, the shortest distance is 1/√6 units.

**iv.**

**Solution:**

Given:

The vector equations are

$$\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - t)\hat{k}$$
$$\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$$

The above equations can be written as

$$\vec{r} = (i - 2j + 3k) + t(-i + j - k)$$
$$\vec{r} = (i - j - k) + s(i + 2j - 2k)$$

By using the formula,

Shortest distance is given by $\left|\frac{(a_2 - a_1)\cdot(b_1 \times b_2)}{|(b_1 \times b_2)|}\right|$

$$(b_1 \times b_2) = -3j - 3k$$
$$(a_2 - a_1) = j - 4k$$
$$(a_2 - a_1)\cdot(b_1 \times b_2) = 9$$
$$|(b_1 \times b_2)| = 3\sqrt{2}$$

Hence,
The shortest distance is $9/3\sqrt{2} = 3/\sqrt{2}$ units.



**v.**
**Solution:**
Given:
Equation of lines is,

$$\vec{r} = (\lambda - 1)\hat{i} + (\lambda + 1)\hat{j} - (1 + \lambda)\hat{k}$$
$$\vec{r} = \left(-\hat{i} + \hat{j} - \hat{k}\right) + \lambda\left(\hat{i} + \hat{j} - \hat{k}\right) \text{ and}$$

$$\vec{r} = (1 - \mu)\hat{i} + (2\mu - 1)\hat{j} + (\mu + 2)\hat{k}$$
$$\vec{r} = \left(\hat{i} - \hat{j} + 2\hat{k}\right) + \mu\left(-\hat{i} + 2\hat{j} + \hat{k}\right)$$

We know that,

Shortest distance between lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by

$$\text{S.D.} = \left|\frac{\left(\vec{a_2} - \vec{a_1}\right).\left(\vec{b_1} \times \vec{b_2}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|}\right| \quad \text{........ (1)}$$

Where,

$$\vec{a_1} = \left(-\hat{i} + \hat{j} - \hat{k}\right), \vec{b_1} = \left(\hat{i} + \hat{j} - \hat{k}\right) \text{ and}$$
$$\vec{a_2} = \left(\hat{i} - \hat{j} + 2\hat{k}\right), \vec{b_2} = \left(-\hat{i} + 2\hat{j} + \hat{k}\right)$$

So now,

$$\left(\vec{a_2}-\vec{a_1}\right) = \left(\hat{i}-\hat{j}+2\hat{k}\right)-\left(-\hat{i}+\hat{j}-\hat{k}\right)$$
$$= \hat{i}-\hat{j}+2\hat{k}+\hat{i}-\hat{j}+\hat{k}$$
$$\left(\vec{a_2}-\vec{a_1}\right) = 2\hat{i}-2\hat{j}+3\hat{k}$$
……. (2)

Let us solve for b, we get

$$\left(\vec{b_1}\times\vec{b_2}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix}$$
$$= \hat{i}\left(1+2\right)-\hat{j}\left(1-1\right)+\hat{k}\left(2+1\right)$$
$$\left(\vec{b_1}\times\vec{b_2}\right) = 3\hat{i}+3\hat{k}$$

By solving (2) and (3), we get

$$\left(\vec{a_2}-\vec{a_1}\right)\cdot\left(\vec{b_1}\times\vec{b_2}\right) = \left(2\hat{i}-2\hat{j}+3\hat{k}\right)\left(3\hat{i}+3\hat{k}\right)$$
$$= (2)(3)+(-2)(0)+(3)(3)$$
$$= 6+0+9$$
$$= 15$$
$$\left|\vec{b_1}\times\vec{b_2}\right| = \sqrt{(3)^2+(3)^2}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

Now let us substitute the above obtained values in equation (1), to get the shortest distance between given lines,

$$\text{S.D.} = \left|\frac{15}{3\sqrt{2}}\right|$$
$$= \frac{5}{\sqrt{2}}$$

Hence, the shortest distance is 5/√2 units.