EXERCISE 19.21

Evaluate the following integrals:

1. $\int \frac{x}{\sqrt{x^2+6x+10}} dx$

Solution:

Given $I = \int \frac{x}{\sqrt{x^2+6x+10}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

$\Rightarrow p x + q = \lambda (2ax + b) + \mu$

$\Rightarrow x = \lambda (2x + 6) + \mu$

$\therefore \lambda = 1/2$ and $\mu = -3$

Let $x = 1/2(2x + 6) - 3$ and split,

$$\Rightarrow \int \frac{x}{\sqrt{x^2+6x+10}} dx = \int \left(\frac{2x+6}{2\sqrt{x^2+6x+10}} - \frac{3}{\sqrt{x^2+6x+10}}\right) dx$$

$$= \int \frac{x+3}{\sqrt{x^2+6x+10}} dx - 3\int \frac{1}{\sqrt{x^2+6x+10}} dx$$

Consider $\int \frac{x+3}{\sqrt{x^2+6x+10}} dx$

Let $u = x^2 + 6x + 10 \rightarrow dx = \frac{1}{2x+6} du$

$$\Rightarrow \int \frac{x+3}{\sqrt{x^2+6x+10}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2 + 6x + 10}$$

Consider $\int \frac{1}{\sqrt{x^2+6x+10}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 6x + 10}} dx = \int \frac{1}{\sqrt{(x+3)^2 + 1}} dx$$

Let u = x + 3 → dx = du

$$\Rightarrow \int \frac{1}{\sqrt{(x+3)^2 + 1}} dx = \int \frac{1}{\sqrt{(u)^2 + 1}} du$$

We know that $\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2 + 1}} du = \sinh^{-1}(u)$$

$$= \sinh^{-1}(x+3)$$

Then,

$$\Rightarrow \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx = \int \frac{x+3}{\sqrt{x^2 + 6x + 10}} dx - 3\int \frac{1}{\sqrt{x^2 + 6x + 10}} dx$$

$$= \sqrt{x^2 + 6x + 10} - 3\sinh^{-1}(x+3) + c$$

$$\therefore I = \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx = \sqrt{x^2 + 6x + 10} - 3\sinh^{-1}(x+3) + c$$

2. $\int \frac{2x+1}{\sqrt{x^2 + 2x - 1}} dx$

Solution:

Given $I = \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

$\Rightarrow$ p x + q = λ (2ax + b) + μ

$\Rightarrow$ 2x + 1 = λ (2x + 2) + μ

∴ λ = 1 and μ = -1

Let 2x + 1 = 2x + 2 – 1 and split,

$$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx = \int \left(\frac{2x+2}{\sqrt{x^2+2x-1}} - \frac{1}{\sqrt{x^2+2x-1}}\right) dx$$

$$= 2\int \frac{x+1}{\sqrt{x^2+2x-1}} dx - \int \frac{1}{\sqrt{x^2+2x-1}} dx$$

Consider $\int \frac{x+1}{\sqrt{x^2+2x-1}} dx$

Let $u = x^2 + 2x - 1 \rightarrow dx = \frac{1}{2x+2} du$

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x-1}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}(2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+2x-1}$$

Consider $\int \frac{1}{\sqrt{x^2+2x-1}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x-1}}dx = \int \frac{1}{\sqrt{(x+1)^2-2}}dx$$

Let $u = \frac{x+1}{\sqrt{2}} \to dx = \sqrt{2}du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2-2}}dx = \int \frac{\sqrt{2}}{\sqrt{2u^2-2}}du$$

$$= \int \frac{1}{\sqrt{u^2-1}}du$$

We know that $\int \frac{1}{\sqrt{x^2-1}}dx = \cosh^{-1}x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1}(u)$$

$$= \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

Then, $\Rightarrow \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx = 2\int \frac{x+1}{\sqrt{x^2+2x-1}}dx - \int \frac{1}{\sqrt{x^2+2x-1}}dx$

$$= 2\sqrt{x^2+2x-1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx = 2\sqrt{x^2+2x-1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

3. $\int \frac{x+1}{\sqrt{4+5x-x^2}}dx$

Solution:

Given $I = \int \frac{x+1}{\sqrt{4+5x-x^2}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

⇒ p x + q = λ (2ax + b) + μ

⇒ x + 1 = λ (-2x + 5) + μ

∴ λ = -1/2 and μ = 7/2

Let x + 1 = – 1/2(–2x + 5) + 7/2

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}}dx = \int \left(\frac{-2x+5}{2\sqrt{-x^2+5x+4}} + \frac{7}{2\sqrt{-x^2+5x+4}}\right)dx$$

$$= \frac{-1}{2}\int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx + \frac{7}{2}\int \frac{1}{\sqrt{-x^2+5x+4}}dx$$

Consider $\int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx$

Let $u = -x^2 + 5x + 4 \rightarrow dx = \frac{1}{-2x+5}du$

$$\Rightarrow \int \frac{-2x+5}{\sqrt{-x^2+5x+4}}dx = -\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow -\int \frac{1}{\sqrt{u}}du = -(2\sqrt{u})$$

$$= -2\sqrt{-x^2+5x+4}$$

Consider $\int \frac{1}{\sqrt{-x^2+5x+4}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2+5x+4}}dx = \int \frac{1}{\sqrt{-\left(x-\frac{5}{2}\right)^2 + \frac{41}{4}}}dx$$

Let $u = \frac{2x-5}{\sqrt{41}} \rightarrow dx = \frac{\sqrt{41}}{2}du$

$$\Rightarrow \int \frac{1}{\sqrt{-\left(x - \frac{5}{2}\right)^2 + \frac{41}{4}}} dx = \int \frac{\sqrt{41}}{\sqrt{41 - 41u^2}} du$$

$$= \int \frac{1}{\sqrt{1 - u^2}} du$$

We know that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1 - u^2}} du = \sin^{-1}\left(\frac{2x - 5}{\sqrt{41}}\right)$$

Then,

$$\Rightarrow \int \frac{x + 1}{\sqrt{-x^2 + 5x + 4}} dx = \frac{1}{2}\int \frac{-2x + 5}{\sqrt{-x^2 + 5x + 4}} dx + \frac{7}{2}\int \frac{1}{\sqrt{-x^2 + 5x + 4}} dx$$

$$= -\sqrt{-x^2 + 5x + 4} + \frac{7}{2}\left(\sin^{-1}\left(\frac{2x - 5}{\sqrt{41}}\right)\right) + c$$

$$\therefore I = \int \frac{x + 1}{\sqrt{-x^2 + 5x + 4}} dx = -\sqrt{-x^2 + 5x + 4} + \frac{7}{2}\left(\sin^{-1}\left(\frac{2x - 5}{\sqrt{41}}\right)\right) + c$$

4. $\int \frac{6x - 5}{\sqrt{3x^2 - 5x + 1}} dx$

Solution:

Given $I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu$

$\Rightarrow p x + q = \lambda (2ax + b) + \mu$

$\Rightarrow 6x – 5 = \lambda (6x – 5) + \mu$

$\therefore \lambda = 1$ and $\mu = 0$

Let $u = 3x^2 - 5x + 1 \rightarrow dx = \frac{1}{6x-5}du$

$$\Rightarrow \int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx = \int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \int \frac{1}{\sqrt{u}}du = \left(2\sqrt{u}\right) + c$$

$$= 2\sqrt{3x^2-5x+1} + c$$

$$\therefore I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx = 2\sqrt{3x^2-5x+1} + c$$

5. $\int \frac{3x+1}{\sqrt{5-2x-x^2}}dx$

Solution:

Given $I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Writing numerator as $px + q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + \mu$

$\Rightarrow px + q = \lambda(2ax + b) + \mu$

$\Rightarrow 3x + 1 = \lambda(-2x - 2) + \mu$

$\therefore \lambda = -3/2$ and $\mu = -2$

Let $3x + 1 = -(3/2)(-2x - 2) - 2$

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = \int \left(\frac{-3(-2x-2)}{2\sqrt{-x^2-2x+5}} - \frac{2}{\sqrt{-x^2-2x+5}}\right)dx$$

$$= 3\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx - 2\int \frac{1}{\sqrt{-x^2-2x+5}}dx$$

Consider $\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx$

Let $u = -x^2 - 2x + 5 \rightarrow dx = \frac{1}{-2x-2}du$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2-2x+5}}dx = \int -\frac{1}{2\sqrt{u}}du$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -(\sqrt{u})$$

$$= -\sqrt{-x^2-2x+5}$$

Consider $\int \frac{1}{\sqrt{-x^2-2x+5}}dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2-2x+5}}dx = \int \frac{1}{\sqrt{6-(x+1)^2}}dx$$

Let $u = \frac{x+1}{\sqrt{6}} \rightarrow dx = \sqrt{6}du$

$$\Rightarrow \int \frac{1}{\sqrt{6-(x+1)^2}}dx = \int \frac{\sqrt{6}}{\sqrt{6-6u^2}}du$$

$$= \int \frac{1}{\sqrt{1-u^2}}du$$

We know that $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$$

Then,

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = 3\int \frac{x+1}{\sqrt{-x^2-2x+5}}dx - 2\int \frac{1}{\sqrt{-x^2-2x+5}}dx$$

$$= -3\sqrt{-x^2-2x+5} - 2\left(\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)\right) + c$$

$$\therefore I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}}dx = -3\sqrt{-x^2-2x+5} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$$