EXERCISE 6.2

Evaluate the following determinant:

$$(i)\begin{vmatrix}1 & 3 & 5\\ 2 & 6 & 10\\ 31 & 11 & 38\end{vmatrix}$$

$$(ii)\begin{vmatrix}67 & 19 & 21\\ 39 & 13 & 14\\ 81 & 24 & 26\end{vmatrix}$$

$$(iii)\begin{vmatrix}a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}$$

$$(iv)\begin{vmatrix}1 & -3 & 2\\ 4 & -1 & 2\\ 3 & 5 & 2\end{vmatrix}$$

$$(v)\begin{vmatrix}1 & 4 & 9\\ 4 & 9 & 16\\ 9 & 16 & 25\end{vmatrix}$$

$$(vi)\begin{vmatrix}6 & 3 & -2\\ 2 & -1 & 2\\ -10 & 5 & 2\end{vmatrix}$$

$$(vii)\begin{vmatrix}1 & 3 & 9 & 27\\ 3 & 9 & 27 & 1\\ 9 & 27 & 1 & 3\\ 27 & 1 & 3 & 9\end{vmatrix}$$

$$(viii)\begin{vmatrix}102 & 18 & 36\\ 1 & 3 & 4\\ 17 & 3 & 6\end{vmatrix}$$

Solution:

(i) Given

$$\begin{vmatrix}1 & 3 & 5\\ 2 & 6 & 10\\ 31 & 11 & 38\end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & 5 \\ 1 & 3 & 5 \\ 31 & 11 & 38 \end{vmatrix}$

Now by applying, $R_2 \rightarrow R_2 - R_1$, we get,

$\Rightarrow \Delta = 2\begin{vmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 31 & 11 & 38 \end{vmatrix} = 0$

So, $\Delta = 0$

(II) Given

$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$

Let, $\Delta = \begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$

By applying column operation $C_1 \rightarrow C_1 - 4C_3$, we get,

$\Rightarrow \Delta = \begin{vmatrix} 4 & 19 & 21 \\ -3 & 13 & 14 \\ -3 & 24 & 26 \end{vmatrix}$

Again by applying row operation, $R_1 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 - R_2$, we get

$\Rightarrow \Delta = \begin{vmatrix} 1 & 32 & 35 \\ -3 & 13 & 14 \\ 0 & 11 & 12 \end{vmatrix}$

Now, applying $R_2 \rightarrow R_2 + 3R_1$, we get,

$\Rightarrow \Delta = \begin{vmatrix} 1 & 32 & 35 \\ 0 & 109 & 119 \\ 0 & 11 & 12 \end{vmatrix}$

= 1[(109) (12) – (119) (11)]

= 1308 – 1309

= – 1

So, $\Delta = -1$

(III) Given,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$= a(bc - f^2) - h(hc - fg) + g(hf - bg)$

$= abc - af^2 - ch^2 + fgh + fgh - bg^2$

$= abc + 2fgh - af^2 - bg^2 - ch^2$

So, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

(iv) Given

$= \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

Let, $\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

By taking 2 as common we get,

$\Rightarrow \Delta = 2\begin{vmatrix} 1 & -3 & 1 \\ 4 & -1 & 1 \\ 3 & 5 & 1 \end{vmatrix}$

Now by applying, row operation $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$\Rightarrow \Delta = 2\begin{vmatrix} 1 & -3 & 1 \\ 3 & 2 & 0 \\ 2 & 8 & 0 \end{vmatrix}$

$= 2[1(24 - 4)] = 40$

So, $\Delta = 40$

(v) Given

$$\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$

By applying column operation $C_3 \to C_3 - C_2$, we get,

$\Rightarrow \Delta = \begin{vmatrix} 1 & 4 & 5 \\ 4 & 9 & 7 \\ 9 & 16 & 9 \end{vmatrix}$

Again by applying column operation $C_2 \to C_2 + C_1$, we get,

$\Rightarrow \Delta = \begin{vmatrix} 1 & 5 & 5 \\ 4 & 13 & 7 \\ 9 & 25 & 9 \end{vmatrix}$

Now by applying $C_2 \to C_2 - 5C_1$ and $C_3 \to C_3 - 5C_1$ we get,

$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -7 & -13 \\ 9 & -20 & -36 \end{vmatrix}$

$= 1[(-7)(-36) - (-20)(-13)]$

$= 252 - 260$

$= -8$

So, $\Delta = -8$

(vi) Given,

$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$

Let, $\Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$

Applying row operations, $R_1 \to R_1 - 3R_2$ and $R_3 \to R_3 + 5R_2$ we get,

$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & -4 \\ 2 & -1 & 2 \\ 0 & 0 & 12 \end{vmatrix} = 0$

So, $\Delta = 0$

(vii) Given

$$\begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix}$$

$$\text{Let } \Delta = \begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 3 & 3^2 & 3^3 & 1 \\ 3^2 & 3^3 & 1 & 3 \\ 3^3 & 1 & 3 & 3^2 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3 + C_4$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1+3+3^2+3^3 & 3 & 3^2 & 3^3 \\ 1+3+3^2+3^3 & 3^2 & 3^3 & 1 \\ 1+3+3^2+3^3 & 3^3 & 1 & 3 \\ 1+3+3^2+3^3 & 1 & 3 & 3^2 \end{vmatrix}$$

$$\Rightarrow \Delta = (1+3+3^2+3^3)\begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 1 & 3^2 & 3^3 & 1 \\ 1 & 3^3 & 1 & 3 \\ 1 & 1 & 3 & 3^2 \end{vmatrix}$$

Now, applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, we get

$$\Rightarrow \Delta = (1+3+3^2+3^3)\begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 0 & 3^2-3 & 3^3-3^2 & 1-3^3 \\ 0 & 3^3-3 & 1-3^2 & 3-3^3 \\ 0 & 1-3 & 3-3^2 & 3^2-3^3 \end{vmatrix}$$

$$\Rightarrow \Delta = (1+3+3^2+3^3)\begin{vmatrix} 6 & 18 & -26 \\ 24 & -8 & -24 \\ -2 & -6 & -18 \end{vmatrix}$$

$$\Rightarrow \Delta = (1+3+3^2+3^3)2^3\begin{vmatrix} 3 & -9 & 13 \\ 12 & 4 & 12 \\ -1 & 3 & 9 \end{vmatrix}$$

Now, applying $R_1 \to R_1 + 3R_3$

$$\Rightarrow \Delta = (1 + 3 + 3^2 + 3^3)2^3 \begin{vmatrix} 0 & 0 & 40 \\ 12 & 4 & 12 \\ -1 & 3 & 9 \end{vmatrix}$$

$$= (1 + 3 + 3^2 + 3^3)2^3[40(36 - (-4))]$$

$= (40)(8)(40)(40) = 512000$

So, $\Delta = 512000$

(viii) Given,

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

$$\Rightarrow \Delta = 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1$, we get,

$$\Rightarrow \Delta = 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

So, $\Delta = 0$

2. Without expanding, show that the value of each of the following determinants is zero:

$$(i) \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 6 & 3 & -2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

$(iii)\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$

$(iv)\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ac \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$

$(v)\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$

$(vi)\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$

$(vii)\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$

$(viii)\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$

$(ix)\begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$

$(x)\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$

$(xi)\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

$(xii)\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$

$(xiii)\begin{vmatrix} sin\alpha & cos\alpha & cos(\alpha+\delta) \\ sin\beta & cos\beta & cos(\beta+\delta) \\ sin\gamma & cos\gamma & cos(\gamma+\delta) \end{vmatrix}$

$$(xiv)\begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

$$(xv)\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$$

$$(xvi)\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

$$(xvii)\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}, where\ A, B, C\ are\ the\ angles\ of\ \Delta ABC$$

Solution:

(i) Given,

$$\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

$$Let\ \Delta = \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

Now by applying row operation $R_3 \to R_3 - R_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 4 & 1 & -2 \end{vmatrix}$$

Again apply row operations $R_2 \to R_2 - R_1$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 8 & 2 & 7 \\ 4 & 1 & -2 \\ 4 & 1 & -2 \end{vmatrix}$$

As, $R_2 = R_3$, therefore the value of the determinant is zero.

(ii) Given,

$$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

Taking (– 2) common from C_1 in above matrix we get,

$$\Rightarrow \Delta = \begin{vmatrix} -3 & -3 & 2 \\ -1 & -1 & 2 \\ 5 & 5 & 2 \end{vmatrix}$$

As, $C_1 = C_2$, hence the value of the determinant is zero.

(iii) Given,

$$\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

Now by applying column operation $C_3 \to C_3 - C_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 2 & 3 & 7 \end{vmatrix}$$

As, $R_1 = R_3$, so value so determinant is zero.

(iv) Given,

$$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix}$$

Multiplying R_1, R_2 and R_3 with a, b and c respectively we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix}$$

Now by taking, abc common from C_3 gives,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix}$$

As, $C_1 = C_3$ hence the value of determinant is zero.

(v) Given,

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$$

Now by applying column operation $C_3 \rightarrow C_3 - C_2$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} a+b & 2a+b & a \\ 2a+b & 3a+b & a \\ 4a+b & 5a+b & a \end{vmatrix}$$

Again applying column operation $C_2 \rightarrow C_2 - C_1$ gives,

$$\Rightarrow \Delta = \begin{vmatrix} a+b & a & a \\ 2a+b & a & a \\ 4a+b & a & a \end{vmatrix}$$

As, $C_2 = C_3$, so the value of the determinant is zero.

(vi) Given,

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 0 & b-a & (a-b)c \\ 0 & c-a & (a-c)b \end{vmatrix}$$

Taking $(b - a)$ and $(c - a)$ common from R_2 and R_3 respectively,

$$\Rightarrow \Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} - (b-a)(c-a)\begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$= [(b - a)(c - a)]\,[(c + a) - (b + a) - (-b + c)]$

$= [(b - a)(c - a)]\,[c + a + b - a - b - c]$

$= [(b - a)(c - a)]\,[0] = 0$

(vii) Given,

$$\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$$

Now by applying column operation, $C_1 \rightarrow C_1 - 8C_3$ we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 2 & 2 & 3 \end{vmatrix}$$

As, $C_1 = C_2$ hence, the determinant is zero.

(viii) Given,

$$\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$

Multiplying C_1, C_2 and C_3 with z, y and x respectively we get,

$\Rightarrow \Delta = \left(\frac{1}{xyz}\right)\begin{vmatrix} 0 & xy & yx \\ -xz & 0 & zx \\ -yz & -zy & 0 \end{vmatrix}$

Now, taking y, x and z common from R_1, R_2 and R_3 gives,

$\Rightarrow \Delta = \left(\frac{1}{xyz}\right)\begin{vmatrix} 0 & x & x \\ -z & 0 & z \\ -y & -y & 0 \end{vmatrix}$

Applying $C_2 \rightarrow C_2 - C_1$ gives,

$\Rightarrow \Delta = \left(\frac{1}{xyz}\right)\begin{vmatrix} 0 & 0 & x \\ -z & -z & z \\ -y & -y & 0 \end{vmatrix}$

As, $C_1 = C_2$, therefore determinant is zero.

(ix) Given,

$$\begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$

Applying $C_2 \rightarrow C_2 - 7C_3$, we get

$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix}$

As, $C_1 = C_2$, hence determinant is zero

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$

Now we have to apply the column operation $C_3 \to C_3 - C_2$, and $C_4 \to C_4 - C_1$, then we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 - 2^2 & 4^2 - 1^2 \\ 2^2 & 3^2 & 4^2 - 3^2 & 5^2 - 2^2 \\ 3^2 & 4^2 & 5^2 - 4^2 & 6^2 - 3^2 \\ 4^2 & 5^2 & 6^2 - 5^2 & 7^2 - 4^2 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1^2 & 2^2 & 5 & 15 \\ 2^2 & 3^2 & 7 & 21 \\ 3^2 & 4^2 & 9 & 27 \\ 4^2 & 5^2 & 11 & 33 \end{vmatrix}$$

Taking 3 common from C_4 we get,

$$\Rightarrow \Delta = 3\begin{vmatrix} 1^2 & 2^2 & 5 & 5 \\ 2^2 & 3^2 & 7 & 7 \\ 3^2 & 4^2 & 9 & 9 \\ 4^2 & 5^2 & 11 & 11 \end{vmatrix}$$

As, C3 = C4 so, the determinant is zero.

(x) Given,

(xi) Given,

$$\begin{vmatrix} a & b & c \\ a + 2x & b + 2y & c + 2z \\ x & y & z \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

Now by applying, $C_2 \to C_2 + C_1$ and $C_3 \to C_3 + C_1$, we get

$$\Rightarrow \Delta = \begin{vmatrix} a & b & c \\ 2a+2x & 2b+2y & 2c+2z \\ a+x & b+y & c+z \end{vmatrix}$$

Taking 2 common from R_2 we get,

$$\Rightarrow \Delta = 2\begin{vmatrix} a & b & c \\ a+x & b+y & c+z \\ a+x & b+y & c+z \end{vmatrix}$$

As, $R_2 = R_3$, hence value of determinant is zero.

(xii) Given,

$$\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 2^{2x}+2^{-2x}+2 & 2^{2x}+2^{-2x}-2 & 1 \\ 3^{2x}+3^{-2x}+2 & 3^{2x}+3^{-2x}-2 & 1 \\ 4^{2x}+4^{-2x}+2 & 4^{2x}+4^{-2x}-2 & 1 \end{vmatrix}$$

By applying, column operation $C_1 \to C_1 - C_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 4 & 2^{2x}+2^{-2x}-2 & 1 \\ 4 & 3^{2x}+3^{-2x}-2 & 1 \\ 4 & 4^{2x}+4^{-2x}-2 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = 4\begin{vmatrix} 1 & 2^{2x}+2^{-2x}-2 & 1 \\ 1 & 3^{2x}+3^{-2x}-2 & 1 \\ 1 & 4^{2x}+4^{-2x}-2 & 1 \end{vmatrix}$$

As $C_1 = C_3$ hence determinant is zero.

(xiii) Given,

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

Multiplying C_1 with $\sin\delta$, C_2 with $\cos\delta$, we get

$$\Rightarrow \Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos(\gamma+\delta) \end{vmatrix}$$

Now, by applying column operation, $C_2 \to C_2 - C_1$, we get,

$$\Rightarrow \Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos\beta\cos\delta - \sin\beta\sin\delta & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta & \cos(\gamma+\delta) \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos(\alpha+\delta) & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos(\beta+\delta) & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos(\gamma+\delta) & \cos(\gamma+\delta) \end{vmatrix}$$

As $C_2 = C_3$ hence determinant is zero.

(xiv) Given,

$$\begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} \sin^2 23^\circ + \sin^2 67^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ - \sin^2 23^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ + \sin^2 23^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

Using, sin (90 – A) = cos A, $\sin^2 A + \cos^2 A = 1$, and $\cos 180^\circ = -1$,

$$\Rightarrow \Delta = \begin{vmatrix} \sin^2 23^\circ + \cos^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -(\sin^2 67^\circ + \cos^2 67^\circ) & -\sin^2 23^\circ & \cos^2 180^\circ \\ -(1 - \sin^2 23^\circ) & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & \sin^2 67^\circ & -1 \\ -1 & -\sin^2 23^\circ & 1 \\ -\cos^2 23^\circ & \sin^2 23^\circ & \cos^2 23^\circ \end{vmatrix}$$

Taking, (– 1) common from C_1, we get

$$\Rightarrow \Delta = -\begin{vmatrix} -1 & \sin^2 67^\circ & -1 \\ 1 & \sin^2 23^\circ & 1 \\ \cos^2 23^\circ & \sin^2 23^\circ & \cos^2 23^\circ \end{vmatrix}$$

Therefore, as $C_1 = C_3$ determinant is zero.

(xv) Given,

$$\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$$

Multiplying R_2 with sin y and R_3 with cos y we get,

$$\Rightarrow \Delta = \frac{1}{\sin y \cos y}\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x \sin y & \cos x \sin y & \sin^2 y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

Now, by applying row operation $R_2 \to R_2 + R_3$, we get,

$$= \frac{1}{\sin y \cos y}\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x \sin y - \cos x \cos y & \cos x \sin y + \sin x \cos y & \sin^2 y - \cos^2 y \\ -\cos x \cos y & \sin x \cos y & -\cos^2 y \end{vmatrix}$$

Taking (– 1) common from R_3, we get

$$= \frac{-1}{\sin y\cos y}\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ -\sin x\sin y + \cos x\cos y & -(\cos x\sin y + \sin x\cos y) & -\sin^2 y + \cos^2 y \\ \cos x\cos y & \sin x\cos y & \cos^2 y \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{-1}{\sin y\cos y}\begin{vmatrix} \cos(x+y) & \sin(x+y) & \cos 2y \\ \cos(x+y) & -\sin(x+y) & \cos 2y \\ -\cos x\cos y & \sin x\cos y & -\cos^2 y \end{vmatrix}$$

As $R_1 = R_2$ hence determinant is zero.

(xvi) Given,

$$\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

$$\text{Let } \Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

Multiplying C_2 with √3 and C_3 with √23 we get,

$$\Rightarrow \Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{15} & \sqrt{115} \\ \sqrt{15}+\sqrt{46} & 5\sqrt{3} & \sqrt{230} \\ 3+\sqrt{115} & \sqrt{45} & 5\sqrt{23} \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5}(\sqrt{3}) & \sqrt{5}(\sqrt{23}) \\ \sqrt{15}+\sqrt{46} & \sqrt{5}(\sqrt{15}) & \sqrt{5}(\sqrt{46}) \\ 3+\sqrt{115} & \sqrt{5}(3) & \sqrt{5}(\sqrt{115}) \end{vmatrix}$$

Now taking √5 common from C_2 and C_3 we get,

$$\Rightarrow \Delta = \sqrt{5}\sqrt{5}\begin{vmatrix} \sqrt{23}+\sqrt{3} & (\sqrt{3}) & (\sqrt{23}) \\ \sqrt{15}+\sqrt{46} & (\sqrt{15}) & (\sqrt{46}) \\ 3+\sqrt{115} & (3) & (\sqrt{115}) \end{vmatrix}$$

Applying $C_2 \to C_2 + C_3$

$$\Rightarrow \Delta = 5\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{23}+\sqrt{3} & (\sqrt{23}) \\ \sqrt{15}+\sqrt{46} & \sqrt{15}+\sqrt{46} & (\sqrt{46}) \\ 3+\sqrt{115} & 3+\sqrt{115} & (\sqrt{115}) \end{vmatrix}$$

As $C_1 = C_2$ hence determinant is zero.

(xvii) Given,

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

Now,

$\Delta = \sin^2 A (\cot B - \cot C) - \cot A (\sin^2 B - \sin^2 C) + 1 (\sin^2 B \cot C - \cot B \sin^2 C$

As A, B and C are angles of a triangle,

$A + B + C = 180°$

$\Delta = \sin^2 A \cot B - \sin^2 A \cot C - \cot A \sin^2 B + \cot A \sin^2 C + \sin^2 B \cot C - \cot B \sin^2 C$

By using formulae, we get

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}, \cos B = \frac{a^2 + c^2 - b^2}{2\,a\,c}, \cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b}$$

$\Delta = 0$

Hence proved.

Evaluate the following (3 – 9):

3. $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

Now by applying column operation $C_2 \to C_2 + C_1$

$$\Rightarrow \Delta = \begin{vmatrix} a & b+c+a & a^2 \\ b & c+a+b & b^2 \\ c & a+b+c & c^2 \end{vmatrix}$$

Taking, (a + b + c) common,

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

Again by applying row operation $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} a & 1 & a^2 \\ b-a & 0 & b^2-a^2 \\ c-a & 0 & c^2-a^2 \end{vmatrix}$$

Taking, (b – c) and (c – a) common,

$$\Rightarrow \Delta = (a+b+c)(b-a)(c-a)\begin{vmatrix} a & 1 & a^2 \\ 1 & 0 & b+a \\ 1 & 0 & c+a \end{vmatrix}$$

= (a + b + c) (b – a) (c – a) (b – c)

So, $\Delta = (a + b + c)(b - a)(c - a)(b - c)$

4. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Now by applying row operation, $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ we get,

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b - a & ca - bc \\ 0 & c - a & ab - bc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & bc \\ 0 & b - a & c(a - b) \\ 0 & c - a & b(a - c) \end{vmatrix}$$

Taking $(a - b)$ and $(a - c)$ common we get,

$$\Rightarrow \Delta = (a - b)(a - c) \begin{vmatrix} 1 & a & bc \\ 0 & -1 & c \\ 0 & -1 & b \end{vmatrix}$$

$= (a - b)(c - a)(b - c)$

So, $\Delta = (a - b)(b - c)(c - a)$

5. $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$$

Applying, $C_1 \to C_1 + C_2 + C_3$, we have,

$$\Rightarrow \Delta = \begin{vmatrix} 3x+\lambda & x & x \\ 3x+\lambda & x+\lambda & x \\ 3x+\lambda & x & x+\lambda \end{vmatrix}$$

Taking, $(3x + \lambda)$ common, we get

$$\Rightarrow \Delta = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 1 & x+\lambda & x \\ 1 & x & x+\lambda \end{vmatrix}$$

Applying, $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get,

$$\Rightarrow \Delta = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$= \lambda^2 (3x + \lambda)$

So, $\Delta = \lambda^2 (3x + \lambda)$

6. $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Now we have to apply column operation, $C_1 \to C_1 + C_2 + C_3$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix}$$

Taking, (a + b + c) we get,

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}$$

Now by applying row operation, $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get,

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & a-b & b-c \\ 0 & c-b & a-c \end{vmatrix}$$

$= (a + b + c) [(a - b)(a - c) - (b - c)(c - b)]$

$= (a + b + c) [a^2 - ac - ab + bc + b^2 + c^2 - 2bc]$

$= (a + b + c) [a^2 + b^2 + c^2 - ac - ab - bc]$

So, $\Delta = (a + b + c) [a^2 + b^2 + c^2 - ac - ab - bc]$

7. $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

Let, $\Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$

Now by applying column operation, $C_1 \to C_1 + C_2 + C_3$, we get,

$$\Rightarrow \Delta = \begin{vmatrix} 2+x & 1 & 1 \\ 2+x & x & 1 \\ 2+x & 1 & x \end{vmatrix}$$

$$\Rightarrow \Delta = (2+x)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

Again by applying row operation, $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get,

$$\Rightarrow \Delta = (2+x)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix}$$

$= (2 + x)(x - 1)^2$

So, $\Delta = (2 + x)(x - 1)^2$

8. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ xz^2 & zy^2 & 0 \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

On simplification we get,

$= 0(0 - y^2z^2) - xy^2(0 - x^2yz^2) + xz^2(x^2y^2z - 0)$

$= 0 + x^3y^3z^3 + x^3y^3z^3$

$= 2x^3y^3z^3$

So, $\Delta = 2x^3y^3z^3$

9. $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

$$\text{Let, } \Delta = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

Now by applying row operation we get $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$

$$\Rightarrow \Delta = \begin{vmatrix} a & -a & 0 \\ x & a+y & z \\ 0 & -a & a \end{vmatrix}$$

Again by applying column operation, $C_2 \to C_2 - C_1$

$$\Rightarrow \Delta = \begin{vmatrix} a & 0 & 0 \\ x & a+x+y & z \\ 0 & -a & a \end{vmatrix}$$

= a [a (a + x + y) + az] + 0 + 0

= a^2 (a + x + y + z)

So, Δ = a^2 (a + x + y + z)

10. $If\ \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix},\ \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix},\ then\ prove\ that\ \Delta + \Delta_1 = 0$

Solution:

$$\text{Let, } \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$$

As $|A| = |A|^T$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix}$$

If any two rows or columns of the determinant are interchanged, then determinant changes its sign

$$\Rightarrow \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} - \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & x^2 - yz \\ 0 & 0 & y^2 - zx \\ 0 & 0 & z^2 - xy \end{vmatrix} = 0$$

So, Δ = 0

Hence the proof

Prove the following identities (11 – 45):

11. $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

Solution:

Given,

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

$$L.H.S = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Apply $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$

Taking (a + b + c) common from C_1 we get,

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

Applying, $R_3 \to R_3 - 2R_1$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

$= (a + b + c) [(b - c) (a + b - 2c) - (c - a) (c + a - 2b)]$

$= a^3 + b^3 + c^3 - 3abc$

As, L.H.S = R.H.S

Hence, the proof.

12. $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

Solution:

Consider,

L.H.S = $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

As $|A| = |A|^T$

So, $\begin{vmatrix} b+c & c+a & a+b \\ a-b & b-c & c-a \\ a & b & c \end{vmatrix}$

If any two rows or columns of the determinant are interchanged, then determinant changes its sign

$-\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

Apply $C_1 \to C_1 + C_2 + C_3$

$= -\begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$

Taking $(a + b + c)$ common from C_1 we get,

$= -(a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$

Applying, $R_3 \to R_3 - 2R_1$

$$= -(a+b+c)\begin{vmatrix}1 & b & c\\ 0 & b-c & c-a\\ 0 & c+a-2b & a+b-2c\end{vmatrix}$$

$= -(a + b + c) [(b - c) (a + b - 2c) - (c - a) (c + a - 2b)]$

$= 3abc - a^2 - b^2 - c^2$

Therefore, L.H.S = R.H.S,

Hence the proof.

13. $\begin{vmatrix}a+b & b+c & c+a\\ b+c & c+a & a+b\\ c+a & a+b & b+c\end{vmatrix} = 2\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}$

Solution:

Given,

$$\begin{vmatrix}a+b & b+c & c+a\\ b+c & c+a & a+b\\ c+a & a+b & b+c\end{vmatrix} = 2\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}$$

L.H.S = $\begin{vmatrix}a+b & b+c & c+a\\ b+c & c+a & a+b\\ c+a & a+b & b+c\end{vmatrix}$

Now by applying, $C_1 \rightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix}2(a+b+c) & b+c & c+a\\ 2(a+b+c) & c+a & a+b\\ 2(a+b+c) & a+b & b+c\end{vmatrix}$$

$$= 2\begin{vmatrix}(a+b+c) & b+c & c+a\\ (a+b+c) & c+a & a+b\\ (a+b+c) & a+b & b+c\end{vmatrix}$$

Again apply, $C_2 \rightarrow C_2 - C_1$, and $C_3 \rightarrow C_3 - C_1$, we have

$$= 2\begin{vmatrix} (a+b+c) & -a & -b \\ (a+b+c) & -b & -c \\ (a+b+c) & -c & -a \end{vmatrix}$$

$$= 2\begin{vmatrix} (a+b+c) & a & b \\ (a+b+c) & b & c \\ (a+b+c) & c & a \end{vmatrix}$$

By expanding, we get

$$= 2\left(\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix} + \begin{vmatrix} a & a & b \\ b & b & c \\ c & c & a \end{vmatrix} + \begin{vmatrix} b & a & b \\ c & b & c \\ a & c & a \end{vmatrix}\right)$$

As in second and third determinant both have same column and its value is zero

Therefore,

$$= 2\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{R.H.S}$$

Hence, the proof.

14. $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix},$$

R.H.S = $2(a + b + c)^3$

Applying $C_1 \to C_1 + C_2 + C_3$, we have

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking, $2(a + b + c)$ common we get,

$$= 2(a+b+C)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Now, applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get,

$$= 2(a+b+C)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

Thus, we have

L.H.S = $2(a + b + c)\,[1(a + b + c)^2]$

$= 2(a + b + c)^3$ = R.H.S

15. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution:

Consider,

L.H.S = $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Now by applying, $R_1 \to R_1 + R_2 + R_3$, we get,

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking (a + b + c) common we get,

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get,

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & b+c+a & 0 \\ 2c & 0 & b+c+a \end{vmatrix}$$

$= (a + b + c)^3$ = R.H.S

Hence, proved.

16. $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

Now by applying, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get,

$$= \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & a^2-b^2 \\ 0 & a-c & a^2-c^2 \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}$$

Again by applying $R_3 \to R_3 - R_2$, we get,

$$= (a-b)(a-c)\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 0 & c-a \end{vmatrix}$$

= (a – b) (a – c) (b – c) = R.H.S

Hence, the proof.

17. $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9(a+b)b^2$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get,

$$= \begin{vmatrix} 3a+3b & 3a+3b & 3a+3b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Taking, (3a + 2b) common we get,

$$= (3a+3b)\begin{vmatrix} 1 & 1 & 1 \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Applying, $C_1 \to C_1 - C_2$ and $C_3 \to C_3 - C_2$, we get,

$= (3a + 3b)\begin{vmatrix} 0 & 1 & 0 \\ 2b & a & b \\ -b & a+2b & -2b \end{vmatrix}$

$= (3a + 3b)b^2\begin{vmatrix} 0 & 1 & 0 \\ 2 & a & 1 \\ -1 & a+2b & -2 \end{vmatrix}$

$= 3(a + b)\, b^2\,(3) = 9(a + b)\, b^2$

= R.H.S

Hence, the proof.

18. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Solution:

Consider,

L.H.S $= \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Now by applying, $R_1 \to a\, R_1$, $R_2 \to b\, R_2$, $R_3 \to c\, R_3$

We get,

$= \left(\frac{1}{abc}\right)\begin{vmatrix} a & a^2 & abc \\ b & b^2 & cab \\ c & c^2 & abc \end{vmatrix}$

$= \left(\frac{abc}{abc}\right)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$

$= -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Hence, the proof.

19. $\begin{vmatrix} z & x & y \\ z^2 & x^2 & y^2 \\ z^4 & x^4 & y^4 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \\ x & y & z \end{vmatrix} = xyz(x-y)(y-z)(z-x)(x+y+z)$

Solution:

Given,

$$\begin{vmatrix} z & x & y \\ z^2 & x^2 & y^2 \\ z^4 & x^4 & y^4 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \\ x & y & z \end{vmatrix}$$
$$= xyz(x-y)(y-z)(z-x)(x+y+z)$$

Consider,

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix}$$

By taking xyz common

$$= xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= xyz \begin{vmatrix} 0 & 1 & 0 \\ x-y & y & z-y \\ x^3-y^3 & y^3 & z^3-y^3 \end{vmatrix}$$

$$= xyz(x-y)(z-y) \begin{vmatrix} 0 & 1 & 0 \\ 1 & y & 1 \\ x^2+y^2+xy & y^3 & z^2+y^2+zy \end{vmatrix}$$

$= -xyz(x-y)(z-y)[z^2+y^2+zy-x^2-y^2-xy]$

$= -xyz(x-y)(z-y)[(z-x)(z+x0+y(z-x)]$

$= -xyz(x-y)(z-y)(z-x)(x+y+z)$

= R.H.S

Hence, the proof.

20. $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$

Solution:

Consider,

$L.H.S = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$

Applying, $C_1 \to C_1 + C_2 - 2C_3$

$= \begin{vmatrix} (b+c)^2 - a^2 - 2bc & a^2 & bc \\ (c+a)^2 - b^2 - 2ca & b^2 & ca \\ (a+b)^2 - c^2 - 2ab & c^2 & ab \end{vmatrix}$

$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix}$

Taking $(a^2+b^2+c^2)$, common, we get,

$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get,

$$= (a^2 + b^2 + c^2)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2 - a^2 & ca - bc \\ 0 & c^2 - a^2 & ab - bc \end{vmatrix}$$

$$= (a^2 + b^2 + c^2)(b - a)(c - a)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b + a & -c \\ 0 & c + a & -b \end{vmatrix}$$

$= (a^2 + b^2 + c^2)\,(b - a)\,(c - a)\,[(b + a)\,(-b) - (-c)\,(c + a)]$

$= (a^2 + b^2 + c^2)\,(a - b)\,(c - a)\,(b - c)\,(a + b + c)$

= R.H.S

Hence, the proof.

21. $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

Now by applying row operation, $R_3 \to R_3 - R_2$

$$= \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)2 & 1 & 0 \end{vmatrix}$$

Again by applying, $R_2 \to R_2 - R_1$

$$= \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)2 & 1 & 0 \\ (a+3)2 & 1 & 0 \end{vmatrix}$$

$= [(2a + 4)\,(1) - (1)\,(2a + 6)]$

= – 2

= R.H.S

Hence, the proof.

22. $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c-(a-b)^2 & ab \end{vmatrix}$$

Applying, $C_2 \to C_2 - 2C_1 - 2C_3$, we get,

$$= \begin{vmatrix} a^2 & a^2-(b-c)^2-2a^2-2bc & bc \\ b^2 & b^2-(c-a)^2a^2-(b-c)^2-2b^2-2ca & ca \\ c^2 & c-(a-b)^2a^2-(b-c)^2-2c^2-2ab & ab \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & -(a^2+b^2+c^2) & bc \\ b^2 & -(a^2+b^2+c^2) & ca \\ c^2 & -(a^2+b^2+c^2) & ab \end{vmatrix}$$

Taking, $-(a^2+b^2+c^2)$ common from C_2 we get,

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2-a^2 & 0 & ca-bc \\ c^2-a^2 & 0 & ab-bc \end{vmatrix}$$

$$= -(a^2 + b^2 + c^2)(a-b)(c-a)\begin{vmatrix} a^2 & 1 & bc \\ -(b+a) & 0 & c \\ c+a & 0 & -b \end{vmatrix}$$

$= -(a^2 + b^2 + c^2)(a-b)(c-a)[(-(b+a))(-b) - (c)(c+a)]$

$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$

= R.H.S

Hence, the proof.

23. $\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2+b^2+c^2)$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Applying, $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & b^2+ca-a^2-bc & b^3-a^3 \\ 0 & c^2+ab-a^2-bc & c^3-a^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & b^2-a^2-c(b-a) & b^3-a^3 \\ 0 & c^2-a^2+b(c-a) & c^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & b+a-c & b^2+a^2+ab \\ 0 & c+a+b & c^2+a^2+ac \end{vmatrix}$$

$= (b-a)(c-a)[\{(b+a-c)\}(c^2+a^2+ac) - (b^2+a^2+ab)(c^2+a^2+ac)]$

$= -(a-b)(c-a)(b-c)(a^2+b^2+c^2)$

= R.H.S

Hence, the proof.

24. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution:

Consider,

$$L.H.S = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking, a, b, c common from C_1, C_2, C_3 respectively we get,

$$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying, $C_1 \rightarrow C_1 + C_2 + C_3$, we get,

$$= abc \begin{vmatrix} 2(a+c) & c & a+c \\ 2(a+b) & b & a \\ 2(b+c) & b+c & c \end{vmatrix}$$

$$= 2abc \begin{vmatrix} (a+c) & c & a+c \\ (a+b) & b & a \\ (b+c) & b+c & c \end{vmatrix}$$

Applying, $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get,

$$= 2abc \begin{vmatrix} (a+c) & -a & 0 \\ (a+b) & -a & -b \\ (b+c) & 0 & -b \end{vmatrix}$$

Applying, $C_1 \rightarrow C_1 + C_2 + C_3$, we get,

$$= 2abc\begin{vmatrix} c & -a & 0 \\ 0 & -a & -b \\ c & 0 & -b \end{vmatrix}$$

Taking c, a, b common from C_1, C_2, C_3 respectively, we get,

$$= 2a^2b^2c^2\begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$

Applying, $R_3 \to R_3 - R_1$, we have

$$= 2a^2b^2c^2\begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$= 2a^2b^2c^2 (2)$

$= 4a^2b^2c^2 = \text{R.H.S}$

Hence, proved.

25. $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4)$

Solution:

Consider,

$$\text{L.H.S} = \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

Applying, $C_1 \to C_1 + C_2 + C_3$, we get,

$$= \begin{vmatrix} 3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4 \end{vmatrix}$$

Taking $(3x + 4)$ common we get,

$$= (3x+4)\begin{vmatrix}1 & x & x\\ 1 & x+4 & x\\ 1 & x & x+4\end{vmatrix}$$

Now by applying, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get,

$$= (3x+4)\begin{vmatrix}1 & x & x\\ 0 & 4 & 0\\ 0 & 0 & 4\end{vmatrix}$$

= 16 (3x + 4)

Hence the proof.

26. $\begin{vmatrix}1 & 1+p & 1+p+q\\ 2 & 3+2p & 4+3p+2q\\ 3 & 6+3p & 10+6p+3q\end{vmatrix} = 1$

Solution:

Let $\Delta = \begin{vmatrix}1 & 1+p & 1+p+q\\ 2 & 3+2p & 4+3p+2q\\ 3 & 6+3p & 10+6p+3q\end{vmatrix}$

We know that the value of a determinant remains same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

Applying $C_2 \to C_2 - pC_1$, we get

$$\Delta = \begin{vmatrix}1 & 1+p-p(1) & 1+p+q\\ 2 & 3+2p-p(2) & 4+3p+2q\\ 3 & 6+3p-p(3) & 10+6p+3q\end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix}1 & 1 & 1+p+q\\ 2 & 3 & 4+3p+2q\\ 3 & 6 & 10+6p+3q\end{vmatrix}$$

Applying $C_3 \to C_3 - qC_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p+q-q(1) \\ 2 & 3 & 4+3p+2q-q(2) \\ 3 & 6 & 10+6p+3q-q(3) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 4+3p \\ 3 & 6 & 10+6p \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - pC_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p-p(1) \\ 2 & 3 & 4+3p-p(3) \\ 3 & 6 & 10+6p-p(6) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1-1 & 1 \\ 2 & 3-2 & 4 \\ 3 & 6-3 & 10 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 3 & 10 \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 - C_2$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 1-1 \\ 2 & 1 & 4-2 \\ 3 & 3 & 10-3 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix}$$

Expanding the determinant along R_1, we have

$\Delta = 1[(1)(7) - (3)(2)] - 0 + 0$

∴ Δ = 7 – 6 = 1

Thus, $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$

Hence the proof.