

## EXERCISE 14.1

1. If  $y = \sin x$  and  $x$  changes from  $\frac{\pi}{2}$  to  $\frac{22}{14}$ , what is the approximate change in  $y$ ?

Given  $y = \sin x$  and  $x$  changes from  $\frac{\pi}{2}$  to  $\frac{22}{14}$ .

Let  $x = \frac{\pi}{2}$  so that  $x + \Delta x = \frac{22}{14}$

$$\Rightarrow \frac{\pi}{2} + \Delta x = \frac{22}{14}$$

$$\therefore \Delta x = \frac{22}{14} - \frac{\pi}{2}$$

On differentiating  $y$  with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x)$$

We know that  $\frac{d}{dx}(\sin x) = \cos x$

$$\therefore \frac{dy}{dx} = \cos x$$

When  $x = \frac{\pi}{2}$ , we have  $\frac{dy}{dx} = \cos\left(\frac{\pi}{2}\right)$ .

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = 0$$

We know that if  $y = f(x)$  and  $\Delta x$  is a small increment in  $x$ , then the corresponding increment in  $y$ ,  $\Delta y = f(x + \Delta x) - f(x)$ , is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right) \Delta x$$

**Solution:**

Here,  $\frac{dy}{dx} = 0$  and  $\Delta x = \frac{22}{14} - \frac{\pi}{2}$

$$\Rightarrow \Delta y = (0) \left( \frac{22}{14} - \frac{\pi}{2} \right)$$

$$\therefore \Delta y = 0$$

Thus, there is approximately no change in  $y$ .

**2. The radius of a sphere shrinks from 10 to 9.8 cm. Find approximately the decrease in its volume.**

**Solution:**

Given the radius of a sphere changes from 10 cm to 9.8 cm.

Let  $x$  be the radius of the sphere and  $\Delta x$  be the change in the value of  $x$ .

Hence, we have  $x = 10$  and  $x + \Delta x = 9.8$

$$\Rightarrow 10 + \Delta x = 9.8$$

$$\Rightarrow \Delta x = 9.8 - 10$$

$$\therefore \Delta x = -0.2$$

The volume of a sphere of radius  $x$  is given by

$$V = \frac{4}{3} \pi x^3$$

On differentiating  $V$  with respect to  $x$ , we get

$$\frac{dV}{dx} = \frac{d}{dx} \left( \frac{4}{3} \pi x^3 \right)$$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3} \frac{d}{dx} (x^3)$$

We know  $\frac{d}{dx} (x^n) = nx^{n-1}$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3} (3x^2)$$

$$\therefore \frac{dV}{dx} = 4\pi x^2$$

When  $x = 10$ , we have  $\frac{dV}{dx} = 4\pi(10)^2$ .

$$\Rightarrow \left(\frac{dV}{dx}\right)_{x=10} = 4\pi \times 100$$

$$\Rightarrow \left(\frac{dV}{dx}\right)_{x=10} = 400\pi$$

We know that if  $y = f(x)$  and  $\Delta x$  is a small increment in  $x$ , then the corresponding increment in  $y$ ,  $\Delta y = f(x + \Delta x) - f(x)$ , is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right) \Delta x$$

$$\text{Here, } \frac{dV}{dx} = 400\pi \text{ and } \Delta x = -0.2$$

$$\Rightarrow \Delta V = (400\pi) (-0.2)$$

$$\therefore \Delta V = -80\pi$$

Thus, the approximate decrease in the volume of the sphere is  $80\pi \text{ cm}^3$ .

**3. A circular metal plate expands under heating so that its radius increases by  $k\%$ . Find the approximate increase in the area of the plate, if the radius of the plate before heating is 10 cm.**

**Solution:**

Given the radius of a circular plate initially is 10 cm and it increases by  $k\%$ .

Let  $x$  be the radius of the circular plate, and  $\Delta x$  is the change in the value of  $x$ .

$$\text{Hence, we have } x = 10 \text{ and } \Delta x = \frac{k}{100} \times 10$$

$$\therefore \Delta x = 0.1k$$

The area of a circular plate of radius  $x$  is given by

$$A = \pi x^2$$

On differentiating  $A$  with respect to  $x$ , we get

$$\frac{dA}{dx} = \frac{d}{dx}(\pi x^2)$$

$$\Rightarrow \frac{dA}{dx} = \pi \frac{d}{dx}(x^2)$$

We know  $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dA}{dx} = \pi(2x)$$

$$\therefore \frac{dA}{dx} = 2\pi x$$

When  $x = 10$ , we have  $\frac{dA}{dx} = 2\pi(10)$ .

$$\Rightarrow \left(\frac{dA}{dx}\right)_{x=10} = 20\pi$$

We know that if  $y = f(x)$  and  $\Delta x$  is a small increment in  $x$ , then the corresponding increment in  $y$ ,  $\Delta y = f(x + \Delta x) - f(x)$ , is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right) \Delta x$$

Here,  $\frac{dA}{dx} = 20\pi$  and  $\Delta x = 0.1k$

$$\Rightarrow \Delta A = (20\pi)(0.1k)$$

$$\therefore \Delta A = 2k\pi$$

Thus, the approximate increase in the area of the circular plate is  $2k\pi \text{ cm}^2$ .

**4. Find the percentage error in calculating the surface area of a cubical box if an error of 1% is made in measuring the lengths of the edges of the cube.**

**Solution:**

Given the error in the measurement of the edge of a cubical box is 1%.

Let  $x$  be the edge of the cubical box, and  $\Delta x$  is the error in the value of  $x$ .

Hence, we have  $\Delta x = \frac{1}{100} \times x$

The percentage error is,

$$\text{Error} = \frac{0.12x^2}{6x^2} \times 100\%$$

$$\Rightarrow \text{Error} = 0.02 \times 100\%$$

$$\therefore \text{Error} = 2\%$$

Thus, the error in calculating the surface area of the cubical box is 2%.

$$\therefore \Delta x = 0.01x$$

The surface area of a cubical box of radius  $x$  is given by

$$S = 6x^2$$

On differentiating  $A$  with respect to  $x$ , we get

$$\frac{dS}{dx} = \frac{d}{dx}(6x^2)$$

$$\Rightarrow \frac{dS}{dx} = 6 \frac{d}{dx}(x^2)$$

$$\text{We know } \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\Rightarrow \frac{dS}{dx} = 6(2x)$$

$$\therefore \frac{dS}{dx} = 12x$$

We know that if  $y = f(x)$  and  $\Delta x$  is a small increment in  $x$ , then the corresponding increment in  $y$ ,  $\Delta y = f(x + \Delta x) - f(x)$ , is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right) \Delta x$$

$$\text{Here, } \frac{dS}{dx} = 12x \text{ and } \Delta x = 0.01x$$

$$\Rightarrow \Delta S = (12x)(0.01x)$$

$$\therefore \Delta S = 0.12x^2$$

5. If there is an error of 0.1% in the measurement of the radius of a sphere, find approximately the percentage error in the calculation of the volume of the sphere.

Given the error in the measurement of the radius of a sphere is 0.1%. Let  $x$  be the radius of the sphere and  $\Delta x$  be the error in the value of  $x$ .

$$\text{Hence, we have } \Delta x = \frac{0.1}{100} \times x$$

$$\therefore \Delta x = 0.001x$$

The volume of a sphere of radius  $x$  is given by

$$V = \frac{4}{3} \pi x^3$$

On differentiating  $V$  with respect to  $x$ , we get

$$\frac{dV}{dx} = \frac{d}{dx} \left( \frac{4}{3} \pi x^3 \right)$$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3} \frac{d}{dx} (x^3)$$

$$\text{We know } \frac{d}{dx} (x^n) = nx^{n-1}$$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3} (3x^2)$$

$$\therefore \frac{dV}{dx} = 4\pi x^2$$

We know that if  $y = f(x)$  and  $\Delta x$  is a small increment in  $x$ , then the corresponding increment in  $y$ ,  $\Delta y = f(x + \Delta x) - f(x)$ , is approximately given as

**Solution:**

$$\Delta y = \left( \frac{dy}{dx} \right) \Delta x$$

$$\text{Here, } \frac{dV}{dx} = 4\pi x^2 \text{ and } \Delta x = 0.001x$$

$$\therefore \Delta V = 0.004\pi x^3$$

The percentage error is,

$$\text{Error} = \frac{0.004\pi x^3}{\frac{4}{3}\pi x^3} \times 100\%$$

$$\Rightarrow \text{Error} = \frac{0.004 \times 3}{4} \times 100\%$$

$$\Rightarrow \text{Error} = 0.003 \times 100\%$$

$$\therefore \text{Error} = 0.3\%$$

Thus, the error in calculating the volume of the sphere is 0.3%.

**6. The pressure  $p$  and the volume  $v$  of a gas are connected by the relation  $pv^{1.4} = \text{const.}$  Find the percentage error in  $p$  corresponding to a decrease of  $\frac{1}{2}\%$  in  $v$ .**

**Solution:**

Given  $pv^{1.4} = \text{constant}$  and the decrease in  $v$  is  $\frac{1}{2}\%$ .

Hence, we have 
$$\Delta v = -\frac{\frac{1}{2}}{100} \times v$$

$$\therefore \Delta v = -0.005v$$

We have  $pv^{1.4} = \text{constant}$

Taking log on both sides, we get

$$\text{Log}(pv^{1.4}) = \text{log}(\text{constant})$$

$$\Rightarrow \text{Log } p + \text{log } v^{1.4} = 0 \quad [\because \text{log}(ab) = \text{log } a + \text{log } b]$$

$$\Rightarrow \text{Log } p + 1.4 \text{log } v = 0 \quad [\because \text{log}(a^m) = m \text{log } a]$$

On differentiating both sides with respect to  $v$ , we get

$$\frac{d}{dp}(\text{log } p) \times \frac{dp}{dv} + \frac{d}{dv}(1.4 \text{log } v) = 0$$

$$\Rightarrow \frac{d}{dp}(\log p) \times \frac{dp}{dv} + 1.4 \frac{d}{dv}(\log v) = 0$$

We know  $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{1}{p} \times \frac{dp}{dv} + 1.4 \times \frac{1}{v} = 0$$

$$\Rightarrow \frac{1}{p} \frac{dp}{dv} + \frac{1.4}{v} = 0$$

$$\Rightarrow \frac{1}{p} \frac{dp}{dv} = -\frac{1.4}{v}$$

$$\therefore \frac{dp}{dv} = -\frac{1.4}{v} p$$

We know that if  $y = f(x)$  and  $\Delta x$  is a small increment in  $x$ , then the corresponding increment in  $y$ ,  $\Delta y = f(x + \Delta x) - f(x)$ , is approximately given as

$$\Delta y = \left(\frac{dy}{dx}\right) \Delta x$$

Here,  $\frac{dp}{dv} = -\frac{1.4}{v} p$  and  $\Delta v = -0.005v$

$$\Rightarrow \Delta p = \left(-\frac{1.4}{v} p\right) (-0.005v)$$

$$\Rightarrow \Delta p = (-1.4p) (-0.005)$$

$$\therefore \Delta p = 0.007p$$

The percentage error is,

$$\text{Error} = \frac{0.007p}{p} \times 100\%$$

$$\Rightarrow \text{Error} = 0.007 \times 100\%$$

$$\therefore \text{Error} = 0.7\%$$

Thus, the error in  $p$  corresponding to the decrease in  $v$  is 0.7%.

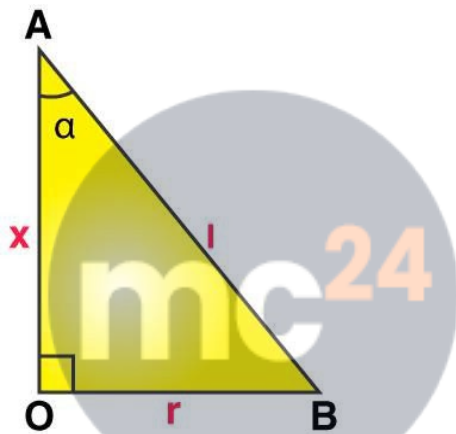
7. The height of a cone increases by  $k\%$ , its semi-vertical angle remaining the same. What is the approximate percentage increase in (i) in total surface area, and (ii) in the volume, assuming that  $k$  is small.

Given the height of a cone increases by  $k\%$ . Let  $x$  be the height of the cone and  $\Delta x$  be the change in the value of  $x$ .

Hence, we have  $\Delta x = \frac{k}{100} \times x$

$$\therefore \Delta x = 0.01kx$$

Let us assume the radius, the slant height and the semi-vertical angle of the cone to be  $r$ ,  $l$  and  $\alpha$  respectively as shown in the figure below.



From the above figure, using trigonometry, we have

$$\tan \alpha = \frac{OB}{OA}$$

$$\Rightarrow \tan \alpha = \frac{r}{x}$$

$$\therefore r = x \tan (\alpha)$$

We also have

$$\cos \alpha = \frac{OA}{AB}$$

$$\Rightarrow \cos \alpha = \frac{x}{l}$$

**Solution:**

$$\Rightarrow l = \frac{x}{\cos \alpha}$$

$$\therefore l = x \sec (\alpha)$$

(i) The total surface area of the cone is given by

$$S = \pi r^2 + \pi r l$$

From above, we have  $r = x \tan (\alpha)$  and  $l = x \sec (\alpha)$ .

$$\Rightarrow S = \pi (x \tan (\alpha))^2 + \pi (x \tan (\alpha)) (x \sec (\alpha))$$

$$\Rightarrow S = \pi x^2 \tan^2 \alpha + \pi x^2 \tan (\alpha) \sec (\alpha)$$

$$\Rightarrow S = \pi x^2 \tan (\alpha) [\tan (\alpha) + \sec (\alpha)]$$

On differentiating  $S$  with respect to  $x$ , we get

$$\frac{dS}{dx} = \frac{d}{dx} [\pi x^2 \tan \alpha (\tan \alpha + \sec \alpha)]$$

$$\Rightarrow \frac{dS}{dx} = \pi \tan \alpha (\tan \alpha + \sec \alpha) \frac{d}{dx} (x^2)$$

We know  $\frac{d}{dx} (x^n) = nx^{n-1}$

$$\Rightarrow \frac{dS}{dx} = \pi \tan \alpha (\tan \alpha + \sec \alpha) (2x)$$

$$\therefore \frac{dS}{dx} = 2\pi x \tan \alpha (\tan \alpha + \sec \alpha)$$

We know that if  $y = f(x)$  and  $\Delta x$  is a small increment in  $x$ , then the corresponding increment in  $y$ ,  $\Delta y = f(x + \Delta x) - f(x)$ , is approximately given as

$$\Delta y = \left( \frac{dy}{dx} \right) \Delta x$$

Here,  $\frac{dS}{dx} = 2\pi x \tan \alpha (\tan \alpha + \sec \alpha)$  and  $\Delta x = 0.01kx$

$$\Rightarrow \Delta S = (2\pi x \tan (\alpha) [\tan (\alpha) + \sec (\alpha)]) (0.01kx)$$

$$\therefore \Delta S = 0.02 k \pi x^2 \tan (\alpha) [\tan (\alpha) + \sec (\alpha)]$$

The percentage increase in  $S$  is,

$$\text{Increase} = \frac{\Delta S}{S} \times 100\%$$

$$\Rightarrow \text{Increase} = \frac{0.02k\pi x^2 \tan \alpha (\tan \alpha + \sec \alpha)}{\pi x^2 \tan \alpha (\tan \alpha + \sec \alpha)} \times 100\%$$

$$\Rightarrow \text{Increase} = 0.02k \times 100\%$$

$$\therefore \text{Increase} = 2k\%$$

Thus, the approximate increase in the total surface area of the cone is  $2k\%$ .

(ii) The volume of the cone is given by

$$V = \frac{1}{3} \pi r^2 x$$

From above, we have  $r = x \tan (\alpha)$ .

$$\Rightarrow V = \frac{1}{3} \pi (x \tan \alpha)^2 x$$

$$\Rightarrow V = \frac{1}{3} \pi (x^2 \tan^2 \alpha) x$$

$$\Rightarrow V = \frac{1}{3} \pi x^3 \tan^2 \alpha$$

On differentiating  $V$  with respect to  $x$ , we get

$$\frac{dV}{dx} = \frac{d}{dx} \left( \frac{1}{3} \pi x^3 \tan^2 \alpha \right)$$

$$\Rightarrow \frac{dV}{dx} = \frac{1}{3} \pi \tan^2 \alpha \frac{d}{dx} (x^3)$$

We know  $\frac{d}{dx} (x^n) = nx^{n-1}$

$$\Rightarrow \frac{dV}{dx} = \frac{1}{3} \pi \tan^2 \alpha (3x^2)$$

$$\therefore \frac{dV}{dx} = \pi x^2 \tan^2 \alpha$$

We know that if  $y = f(x)$  and  $\Delta x$  is a small increment in  $x$ , then the corresponding increment in  $y$ ,  $\Delta y = f(x + \Delta x) - f(x)$ , is approximately given as

$$\Delta y = \left( \frac{dy}{dx} \right) \Delta x$$

Here,  $\frac{dV}{dx} = \pi x^2 \tan^2 \alpha$  and  $\Delta x = 0.01kx$

$$\Rightarrow \Delta V = (\pi x^2 \tan^2 \alpha) (0.01kx)$$

$$\therefore \Delta V = 0.01k\pi x^3 \tan^2 \alpha$$

The percentage increase in  $V$  is,

$$\text{Increase} = \frac{\Delta V}{V} \times 100\%$$

$$\Rightarrow \text{Increase} = \frac{0.01k\pi x^3 \tan^2 \alpha}{\frac{1}{3}\pi x^3 \tan^2 \alpha} \times 100\%$$

$$\Rightarrow \text{Increase} = \frac{0.01k}{\frac{1}{3}} \times 100\%$$

$$\Rightarrow \text{Increase} = 0.03k \times 100\%$$

$$\therefore \text{Increase} = 3k\%$$

Thus, the approximate increase in the volume of the cone is  $3k\%$ .

**8. Show that the relative error in computing the volume of a sphere, due to an error in measuring the radius, is approximately equal to three times the relative error in the radius.**

**Solution:**

Let the error in measuring the radius of a sphere be  $k\%$ .

Let  $x$  be the radius of the sphere and  $\Delta x$  be the error in the value of  $x$

Hence, we have  $\Delta x = \frac{k}{100} \times x$

$$\therefore \Delta x = 0.01kx$$

The volume of a sphere of radius  $x$  is given by

$$V = \frac{4}{3}\pi x^3$$

On differentiating  $V$  with respect to  $x$ , we get

$$\frac{dV}{dx} = \frac{d}{dx} \left( \frac{4}{3}\pi x^3 \right)$$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3} \frac{d}{dx} (x^3)$$

$$\text{We know } \frac{d}{dx} (x^n) = nx^{n-1}$$

$$\Rightarrow \frac{dV}{dx} = \frac{4\pi}{3} (3x^2)$$

$$\therefore \frac{dV}{dx} = 4\pi x^2$$

We know that if  $y = f(x)$  and  $\Delta x$  is a small increment in  $x$ , then the corresponding increment in  $y$ ,  $\Delta y = f(x + \Delta x) - f(x)$ , is approximately given as

$$\Delta y = \left( \frac{dy}{dx} \right) \Delta x$$

$$\text{Here, } \frac{dV}{dx} = 4\pi x^2 \text{ and } \Delta x = 0.01kx$$

$$\Rightarrow \Delta V = (4\pi x^2) (0.01kx)$$

$$\therefore \Delta V = 0.04k\pi x^3$$

The percentage error is,

$$\text{Error} = \frac{0.04k\pi x^3}{\frac{4}{3}\pi x^3} \times 100\%$$

$$\Rightarrow \text{Error} = \frac{0.04k \times 3}{4} \times 100\%$$

$$\Rightarrow \text{Error} = 0.03k \times 100\%$$

$$\therefore \text{Error} = 3k\%$$



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