EXERCISE 2.1

1. Give an example of a function
(i) Which is one-one but not onto.
(ii) Which is not one-one but onto.
(iii) Which is neither one-one nor onto.

Solution:
(i) Let f: Z → Z given by f(x) = 3x + 2
Let us check one-one condition on f(x) = 3x + 2
Injectivity:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y).
f (x) = f(y)
⇒ 3x + 2 =3y + 2
⇒ 3x = 3y
⇒ x = y
⇒ f(x) = f(y)
⇒ x = y
So, f is one-one.
Surjectivity:
Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z(domain).
Let f(x) = y
⇒ 3x + 2 = y
⇒ 3x = y - 2
⇒ x = (y - 2)/3. It may not be in the domain (Z)
Because if we take y = 3,
x = (y - 2)/3 = (3-2)/3 = 1/3 ∉ domain Z.
So, for every element in the co domain there need not be any element in the domain such that f(x) = y.
Thus, f is not onto.

(ii) Example for the function which is not one-one but onto
Let f: Z → N ∪ {0} given by f(x) = |x|
Injectivity:
Let x and y be any two elements in the domain (Z),

Such that f(x) = f(y).
⇒ |x| = |y|
⇒ x = ± y
So, different elements of domain f may give the same image.
So, f is not one-one.
Surjectivity:
Let y be any element in the co domain (Z), such that f(x) = y for some element x in Z (domain).
f(x) = y
⇒ |x| = y
⇒ x = ± y
Which is an element in Z (domain).
So, for every element in the co-domain, there exists a pre-image in the domain.
Thus, f is onto.

(iii) Example for the function which is neither one-one nor onto.
Let f: Z → Z given by f(x) = $2x^2 + 1$
Injectivity:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y).
f(x) = f(y)
⇒ $2x^2+1 = 2y^2+1$
⇒ $2x^2 = 2y^2$
⇒ $x^2 = y^2$
⇒ x = ± y
So, different elements of domain f may give the same image.
Thus, f is not one-one.
Surjectivity:
Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z (domain).
f (x) = y
⇒ $2x^2+1=y$
⇒ $2x^2= y - 1$
⇒ $x^2 = (y-1)/2$
⇒ x = √ ((y-1)/2) ∉ Z always.
For example, if we take, y = 4,
x = ± √ ((y-1)/2)
= ± √ ((4-1)/2)

$= \pm \sqrt{(3/2)} \notin Z$
So, x may not be in Z (domain).
Thus, f is not onto.

2. Which of the following functions from A to B are one-one and onto?
(i) f_1 = {(1, 3), (2, 5), (3, 7)}; A = {1, 2, 3}, B = {3, 5, 7}
(ii) f_2 = {(2, a), (3, b), (4, c)}; A = {2, 3, 4}, B = {a, b, c}
(iii) f_3 = {(a, x), (b, x), (c, z), (d, z)}; A = {a, b, c, d,}, B = {x, y, z}.

Solution:
(i) Consider f_1 = {(1, 3), (2, 5), (3, 7)}; A = {1, 2, 3}, B = {3, 5, 7}
Injectivity:
$f_1(1) = 3$
$f_1(2) = 5$
$f_1(3) = 7$
⇒ Every element of A has different images in B.
So, f_1 is one-one.
Surjectivity:
Co-domain of f_1 = {3, 5, 7}
Range of f_1 =set of images = {3, 5, 7}
⇒ Co-domain = range
So, f_1 is onto.

(ii) Consider f_2 = {(2, a), (3, b), (4, c)}; A = {2, 3, 4}, B = {a, b, c}
Injectivity:
$f_2(2) = a$
$f_2(3) = b$
$f_2(4) = c$
⇒ Every element of A has different images in B.
So, f_2 is one-one.
Surjectivity:
Co-domain of f_2 = {a, b, c}
Range of f_2 = set of images = {a, b, c}
⇒ Co-domain = range
So, f_2 is onto.

(iii) Consider f_3 = {(a, x), (b, x), (c, z), (d, z)} ; A = {a, b, c, d,}, B = {x, y, z}

Injectivity:
f_3 (a) = x
f_3 (b) = x
f_3 (c) = z
f_3 (d) = z
⇒ a and b have the same image x.
Also c and d have the same image z
So, f_3 is not one-one.
Surjectivity:
Co-domain of f_3 ={x, y, z}
Range of f_3 =set of images = {x, z}
So, the co-domain is not same as the range.
So, f_3 is not onto.

3. Prove that the function f: N → N, defined by f(x) = $x^2 + x + 1$, is one-one but not onto

Solution:
Given f: N → N, defined by f(x) = $x^2 + x + 1$
Now we have to prove that given function is one-one
Injectivity:
Let x and y be any two elements in the domain (N), such that f(x) = f(y).
⇒ $x^2 + x + 1 = y^2 + y + 1$
⇒ $(x^2 - y^2) + (x - y) = 0$ `
⇒ (x + y) (x- y) + (x - y) = 0
⇒ (x - y) (x + y + 1) = 0
⇒ x - y = 0 [x + y + 1 cannot be zero because x and y are natural numbers
⇒ x = y
So, f is one-one.
Surjectivity:
When x = 1
$x^2 + x + 1 = 1 + 1 + 1 = 3$
⇒ $x^2 + x + 1 \geq 3$, for every x in N.
⇒ f(x) will not assume the values 1 and 2.
So, f is not onto.

4. Let A = {−1, 0, 1} and f = {(x, x^2) : x ∈ A}. Show that f : A → A is neither one-one nor onto.

Solution:

Given A = {−1, 0, 1} and f = {(x, x^2): x ∈ A}

Also given that, f(x) = x^2

Now we have to prove that given function neither one-one or nor onto.

Injectivity:

Let x = 1

Therefore f(1) = 1^2=1 and

f(-1)=$(-1)^2$=1

⇒ 1 and -1 have the same images.

So, f is not one-one.

Surjectivity:

Co-domain of f = {-1, 0, 1}

f(1) = 1^2 = 1,

f(-1) = $(-1)^2$ = 1 and

f(0) = 0

⇒ Range of f = {0, 1}

So, both are not same.

Hence, f is not onto

5. Classify the following function as injection, surjection or bijection:

(i) f: N → N given by f(x) = x^2

(ii) f: Z → Z given by f(x) = x^2

(iii) f: N → N given by f(x) = x^3

(iv) f: Z → Z given by f(x) = x^3

(v) f: R → R, defined by f(x) = |x|

(vi) f: Z → Z, defined by f(x) = x^2 + x

(vii) f: Z → Z, defined by f(x) = x − 5

(viii) f: R → R, defined by f(x) = sin x

(ix) f: R → R, defined by f(x) = x^3 + 1

(x) f: R → R, defined by f(x) = x^3 − x

(xi) f: R → R, defined by f(x) = $\sin^2$x + $\cos^2$x

(xii) f: Q − {3} → Q, defined by f (x) = (2x +3)/(x-3)

(xiii) f: Q → Q, defined by f(x) = x^3 + 1

(xiv) f: R → R, defined by f(x) = 5x^3 + 4

(xv) f: R → R, defined by f(x) = 5x^3 + 4

(xvi) f: R → R, defined by f(x) = 1 + x^2

(xvii) f: R → R, defined by f(x) = x/(x^2 + 1)

Solution:

(i) Given f: N → N, given by f(x) = x^2

Now we have to check for the given function is injection, surjection and bijection condition.

Injection condition:

Let x and y be any two elements in the domain (N), such that f(x) = f(y).

f(x) = f(y)

$x^2 = y^2$

x = y (We do not get ± because x and y are in N that is natural numbers)

So, f is an injection.

Surjection condition:

Let y be any element in the co-domain (N), such that f(x) = y for some element x in N (domain).

f(x) = y

$x^2 = y$

x = √y, which may not be in N.

For example, if y = 3,

x = √3 is not in N.

So, f is not a surjection.

Also f is not a bijection.

(ii) Given f: Z → Z, given by f(x) = x^2

Now we have to check for the given function is injection, surjection and bijection condition.

Injection condition:

Let x and y be any two elements in the domain (Z), such that f(x) = f(y).

f(x) = f(y)

$x^2 = y^2$

x = ±y

So, f is not an injection.

Surjection test:

Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z (domain).

f(x) = y

$x^2 = y$

$x = \pm \sqrt{y}$ which may not be in Z.
For example, if y = 3,
$x = \pm \sqrt{3}$ is not in Z.
So, f is not a surjection.
Also f is not bijection.

(iii) Given f: N → N given by $f(x) = x^3$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
Let x and y be any two elements in the domain (N), such that f(x) = f(y).
f(x) = f(y)
$x^3 = y^3$
x = y
So, f is an injection
Surjection condition:
Let y be any element in the co-domain (N), such that f(x) = y for some element x in N (domain).
f(x) = y
$x^3 = y$
$x = \sqrt[3]{y}$ which may not be in N.
For example, if y = 3,
$X = \sqrt[3]{3}$ is not in N.
So, f is not a surjection and f is not a bijection.

(iv) Given f: Z → Z given by $f(x) = x^3$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y)
f(x) = f(y)
$x^3 = y^3$
x = y
So, f is an injection.
Surjection condition:
Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z (domain).

f(x) = y
$x^3 = y$
$x = \sqrt[3]{y}$ which may not be in Z.
For example, if y = 3,
$x = \sqrt[3]{3}$ is not in Z.
So, f is not a surjection and f is not a bijection.

(v) Given f: R → R, defined by f(x) = |x|
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y)
f(x) = f(y)
|x|=|y|
x = ±y
So, f is not an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
|x|=y
x = ± y ∈ Z
So, f is a surjection and f is not a bijection.

(vi) Given f: Z → Z, defined by $f(x) = x^2 + x$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y).
f(x) = f(y)
$x^2 + x = y^2 + y$
Here, we cannot say that x = y.
For example, x = 2 and y = - 3
Then,
$x^2 + x = 2^2 + 2 = 6$
$y^2 + y = (-3)^2 - 3 = 6$
So, we have two numbers 2 and -3 in the domain Z whose image is same as 6.

So, f is not an injection.
Surjection test:
Let y be any element in the co-domain (Z),
such that f(x) = y for some element x in Z (domain).
f(x) = y
$x^2 + x = y$
Here, we cannot say $x \in Z$.
For example, y = - 4.
$x^2 + x = -4$
$x^2 + x + 4 = 0$
$x = (-1 \pm \sqrt{-5})/2 = (-1 \pm i\sqrt{5})/2$ which is not in Z.
So, f is not a surjection and f is not a bijection.

(vii) Given f: Z → Z, defined by f(x) = x – 5
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (Z), such that f(x) = f(y).
f(x) = f(y)
x - 5 = y - 5
x = y
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (Z), such that f(x) = y for some element x in Z (domain).
f(x) = y
x - 5 = y
x = y + 5, which is in Z.
So, f is a surjection and f is a bijection

(viii) Given f: R → R, defined by f(x) = sin x
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
Sin x = sin y

Here, x may not be equal to y because sin 0 = sin π.
So, 0 and π have the same image 0.
So, f is not an injection.
Surjection test:
Range of f = [-1, 1]
Co-domain of f = R
Both are not same.
So, f is not a surjection and f is not a bijection.

(ix) Given f: R → R, defined by $f(x) = x^3 + 1$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
$x^3+1 = y^3+ 1$
$x^3= y^3$
x = y
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
$x^3+1=y$
$x = \sqrt[3]{(y - 1)} \in R$
So, f is a surjection.
So, f is a bijection.

(x) Given f: R → R, defined by $f(x) = x^3 – x$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
$x^3 – x = y^3 – y$
Here, we cannot say x = y.
For example, x = 1 and y = -1

$x^3 - x = 1 - 1 = 0$
$y^3 - y = (-1)^3 - (-1) - 1 + 1 = 0$
So, 1 and -1 have the same image 0.
So, f is not an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
$x^3 - x = y$
By observation we can say that there exist some x in R, such that $x^3 - x = y$.
So, f is a surjection and f is not a bijection.

(xi) Given f: R → R, defined by $f(x) = \sin^2 x + \cos^2 x$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
$f(x) = \sin^2 x + \cos^2 x$
We know that $\sin^2 x + \cos^2 x = 1$
So, f(x) = 1 for every x in R.
So, for all elements in the domain, the image is 1.
So, f is not an injection.
Surjection condition:
Range of f = {1}
Co-domain of f = R
Both are not same.
So, f is not a surjection and f is not a bijection.

(xii) Given f: Q − {3} → Q, defined by f (x) = (2x +3)/(x-3)
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (Q − {3}), such that f(x) = f(y).
f(x) = f(y)
$(2x + 3)/(x - 3) = (2y + 3)/(y - 3)$
$(2x + 3)(y - 3) = (2y + 3)(x - 3)$
$2xy - 6x + 3y - 9 = 2xy - 6y + 3x - 9$
$9x = 9y$

x = y
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (Q − {3}), such that f(x) = y for some element x in Q (domain).
f(x) = y
(2x + 3)/(x - 3) = y
2x + 3 = x y − 3y
2x – x y = −3y − 3
x (2−y) = −3 (y + 1)
x = -3(y + 1)/(2 - y) which is not defined at y = 2.
So, f is not a surjection and f is not a bijection.

(xiii) Given f: Q → Q, defined by $f(x) = x^3 + 1$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection test:
Let x and y be any two elements in the domain (Q), such that f(x) = f(y).
f(x) = f(y)
$x^3 + 1 = y^3 + 1$
$x^3 = y^3$
x = y
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (Q), such that f(x) = y for some element x in Q (domain).
f(x) = y
$x^3 + 1 = y$
$x = \sqrt[3]{(y-1)}$, which may not be in Q.
For example, if y= 8,
$x^3 + 1 = 8$
$x^3 = 7$
$x = \sqrt[3]{7}$, which is not in Q.
So, f is not a surjection and f is not a bijection.

(xiv) Given f: R → R, defined by $f(x) = 5x^3 + 4$
Now we have to check for the given function is injection, surjection and bijection

condition.
Injection test:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
$5x^3 + 4 = 5y^3 + 4$
$5x^3 = 5y^3$
$x^3 = y^3$
x = y
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
$5x^3 + 4 = y$
$x^3 = (y - 4)/5 \in R$
So, f is a surjection and f is a bijection.

(xv) Given f: R → R, defined by $f(x) = 5x^3 + 4$
Now we have to check for the given function is injection, surjection and bijection condition.
Injection condition:
Let x and y be any two elements in the domain (R), such that f(x) = f(y).
f(x) = f(y)
$5x^3 + 4 = 5y^3 + 4$
$5x^3 = 5y^3$
$x^3 = y^3$
x = y
So, f is an injection.
Surjection test:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).
f(x) = y
$5x^3 + 4 = y$
$x^3 = (y - 4)/5 \in R$
So, f is a surjection and f is a bijection.

(xvi) Given f: R → R, defined by $f(x) = 1 + x^2$

Now we have to check for the given function is injection, surjection and bijection condition.

Injection condition:

Let x and y be any two elements in the domain (R), such that f(x) = f(y).

f(x) = f(y)

$1 + x^2 = 1 + y^2$

$x^2 = y^2$

x = ± y

So, f is not an injection.

Surjection test:

Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).

f(x) = y

$1 + x^2 = y$

$x^2 = y - 1$

x = ± √-1 = ± i` is not in R.

So, f is not a surjection and f is not a bijection.

(xvii) Given f: R → R, defined by $f(x) = x/(x^2 + 1)$

Now we have to check for the given function is injection, surjection and bijection condition.

Injection condition:

Let x and y be any two elements in the domain (R), such that f(x) = f(y).

f(x) = f(y)

$x /(x^2 + 1) = y /(y^2 + 1)$

$x y^2 + x = x^2 y + y$

$xy^2 - x^2 y + x - y = 0$

$-x y (-y + x) + 1 (x - y) = 0$

$(x - y) (1 - x y) = 0$

x = y or x = 1/y

So, f is not an injection.

Surjection test:

Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain).

f(x) = y

$x /(x^2 + 1) = y$

$y x^2 - x + y = 0$

x = (-(-1) ± √ (1-4y^2))/(2y) if y ≠ 0
= (1 ± √ (1-4y^2))/ (2y), which may not be in R
For example, if y=1, then
(1 ± √ (1-4)) / (2y) = (1 ± i √3)/2, which is not in R
So, f is not surjection and f is not bijection.

6. If f: A → B is an injection, such that range of f = {a}, determine the number of elements in A.

Solution:
Given f: A → B is an injection
And also given that range of f = {a}
So, the number of images of f = 1
Since, f is an injection, there will be exactly one image for each element of f .
So, number of elements in A = 1.

7. Show that the function f: R − {3} → R − {2} given by f(x) = (x-2)/(x-3) is a bijection.

Solution:
Given that f: R − {3} → R − {2} given by f (x) = (x-2)/(x-3)
Now we have to show that the given function is one-one and on-to
Injectivity:
Let x and y be any two elements in the domain (R − {3}), such that f(x) = f(y).
f(x) = f(y)
⇒ (x - 2) /(x - 3) = (y - 2) /(y - 3)
⇒ (x - 2) (y - 3) = (y - 2) (x - 3)
⇒ x y – 3 x – 2 y + 6 = x y - 3y - 2x + 6
⇒ x = y
So, f is one-one.
Surjectivity:
Let y be any element in the co-domain (R − {2}), such that f(x) = y for some element x in R − {3} (domain).
f(x) = y
⇒ (x - 2) /(x - 3) = y
⇒ x - 2 = x y - 3y
⇒ x y - x = 3y - 2
⇒ x (y - 1) = 3y - 2

⇒ x = (3y - 2)/ (y - 1), which is in R - {3}
So, for every element in the co-domain, there exists some pre-image in the domain.
⇒ f is onto.
Since, f is both one-one and onto, it is a bijection.

8. Let *A* = [-1, 1]. Then, discuss whether the following function from *A* to itself is one-one, onto or bijective:
(i) f (x) = x/2
(ii) g (x) = |x|
(iii) h (x) = x^2

Solution:
(i) Given f: A → A, given by f (x) = x/2
Now we have to show that the given function is one-one and on-to
Injection test:
Let x and y be any two elements in the domain (A), such that f(x) = f(y).
f(x) = f(y)
x/2 = y/2
x = y
So, f is one-one.
Surjection test:
Let y be any element in the co-domain (A), such that f(x) = y for some element x in A (domain)
f(x) = y
x/2 = y
x = 2y, which may not be in A.
For example, if y = 1, then
x = 2, which is not in A.
So, f is not onto.
So, f is not bijective.

(ii) Given g: A → A, given by g (x) = |x|
Now we have to show that the given function is one-one and on-to
Injection test:
Let x and y be any two elements in the domain (A), such that f(x) = f(y).
g(x) = g(y)
|x| = |y|

$x = \pm y$
So, f is not one-one.
Surjection test:
For y = -1, there is no value of x in A.
So, g is not onto.
So, g is not bijective.

(iii) Given h: A → A, given by h (x) = x^2
Now we have to show that the given function is one-one and on-to
Injection test:
Let x and y be any two elements in the domain (A), such that h(x) = h(y).
h(x) = h(y)
$x^2 = y^2$
$x = \pm y$
So, f is not one-one.
Surjection test:
For y = - 1, there is no value of x in A.
So, h is not onto.
So, h is not bijective.

9. Are the following set of ordered pair of a function? If so, examine whether the mapping is injective or surjective:
(i) {(x, y): x is a person, y is the mother of x}
(ii) {(a, b): a is a person, b is an ancestor of a}

Solution:
Let f = {(x, y): x is a person, y is the mother of x}
As, for each element x in domain set, there is a unique related element y in co-domain set.
So, f is the function.
Injection test:
As, y can be mother of two or more persons
So, f is not injective.
Surjection test:
For every mother y defined by (x, y), there exists a person x for whom y is mother.
So, f is surjective.
Therefore, f is surjective function.

(ii) Let g = {(a, b): a is a person, b is an ancestor of a}
Since, the ordered map (a, b) does not map 'a' - a person to a living person.
So, g is not a function.

10. Let *A* = {1, 2, 3}. Write all one-one from *A* to itself.

Solution:
Given A = {1, 2, 3}
Number of elements in A = 3
Number of one-one functions = number of ways of arranging 3 elements = 3! = 6
(i) {(1, 1), (2, 2), (3, 3)}
(ii) {(1, 1), (2, 3), (3, 2)}
(iii) {(1, 2), (2, 2), (3, 3)}
(iv) {(1, 2), (2, 1), (3, 3)}
(v) {(1, 3), (2, 2), (3, 1)}
(vi) {(1, 3), (2, 1), (3,2)}

11. If f: R → R be the function defined by $f(x) = 4x^3 + 7$, show that f is a bijection.

Solution:
Given f: R → R is a function defined by $f(x) = 4x^3 + 7$
Injectivity:
Let x and y be any two elements in the domain (R), such that f(x) = f(y)
$\Rightarrow 4x^3 + 7 = 4y^3 + 7$
$\Rightarrow 4x^3 = 4y^3$
$\Rightarrow x^3 = y^3$
$\Rightarrow x = y$
So, f is one-one.
Surjectivity:
Let y be any element in the co-domain (R), such that f(x) = y for some element x in R (domain)
f(x) = y
$\Rightarrow 4x^3 + 7 = y$
$\Rightarrow 4x^3 = y - 7$
$\Rightarrow x^3 = (y - 7)/4$
$\Rightarrow x = \sqrt[3]{(y-7)/4}$ in R

So, for every element in the co-domain, there exists some pre-image in the domain. f is onto.
Since, f is both one-to-one and onto, it is a bijection.