EXERCISE 2.3

Find fog and gof, if

(i) f (x) = e^x, g (x) = $\log_e x$
(ii) f (x) = x^2, g (x) = cos x
(iii) f (x) = |x|, g (x) = sin x
(iv) f (x) = x+1, g(x) = e^x
(v) f (x) = $\sin^{-1} x$, g(x) = x^2
(vi) f (x) = x+1, g (x) = sin x
(vii) f(x)= x + 1, g (x) = 2x + 3
(viii) f(x) = c, c ∈ R, g(x) = sin x^2
(ix) f(x) = x^2 + 2 , g (x) = 1 − 1/ (1-x)

Solution:

(i) Given f (x) = e^x, g(x) = $\log_e x$
Let f: R → (0, ∞); and g: (0, ∞) → R
Now we have to calculate fog,
Clearly, the range of g is a subset of the domain of f.
fog: (0, ∞) → R
(fog) (x) = f (g (x))
= f ($\log_e x$)
= $\log_e e^x$
= x
Now we have to calculate gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R→ R
(gof) (x) = g (f (x))
= g (e^x)
= $\log_e e^x$
= x

(ii) f (x) = x^2, g(x) = cos x
f: R→ [0, ∞) ; g: R→[−1, 1]
Now we have to calculate fog,
Clearly, the range of g is not a subset of the domain of f.
⇒ Domain (fog) = {x: x ∈ domain of g and g (x) ∈ domain of f}
⇒ Domain (fog) = x: x ∈ R and cos x ∈ R}

⇒ Domain of (fog) = R
(fog): R→ R
(fog) (x) = f (g (x))
= f (cos x)
= $\cos^2 x$
Now we have to calculate gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R→R
(gof) (x) = g (f (x))
= g (x^2)
= cos x^2

(iii) Given f (x) = |x|, g(x) = sin x
f: R → (0, ∞) ; g : R→[−1, 1]
Now we have to calculate fog,
Clearly, the range of g is a subset of the domain of f.
⇒ fog: R→R
(fog) (x) = f (g (x))
= f (sin x)
= |sin x|
Now we have to calculate gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog : R→ R
(gof) (x) = g (f (x))
= g (|x|)
= sin |x|

(iv) Given f (x) = x + 1, g(x) = e^x
f: R→R ; g: R → [1, ∞)
Now we have calculate fog:
Clearly, range of g is a subset of domain of f.
⇒ fog: R→R
(fog) (x) = f (g (x))
= f (e^x)
= $e^x + 1$
Now we have to compute gof,
Clearly, range of f is a subset of domain of g.

⇒ fog: R→R
(gof) (x) = g (f (x))
= g (x+1)
= e^{x+1}

(v) Given f (x) = $\sin^{-1} x$, g(x) = x^2
f: [−1,1]→ [(-π)/2 ,π/2]; g : R → [0, ∞)
Now we have to compute fog:
Clearly, the range of g is not a subset of the domain of f.
Domain (fog) = {x: x ∈ domain of g and g (x) ∈ domain of f}
Domain (fog) = {x: x ∈ R and x^2 ∈ [−1, 1]}
Domain (fog) = {x: x ∈ R and x ∈ [−1, 1]}
Domain of (fog) = [−1, 1]
fog: [−1,1] → R
(fog) (x) = f (g (x))
= f (x^2)
= $\sin^{-1} (x^2)$
Now we have to compute gof:
Clearly, the range of f is a subset of the domain of g.
fog: [−1, 1] → R
(gof) (x) = g (f (x))
= g ($\sin^{-1} x$)
= $(\sin^{-1} x)^2$

(vi) Given f(x) = x+1, g(x) = sin x
f: R→R ; g: R→[−1, 1]
Now we have to compute fog
Clearly, the range of g is a subset of the domain of f.
Set of the domain of f.
⇒ fog: R→ R
(fog) (x) = f (g (x))
= f (sin x)
= sin x + 1
Now we have to compute gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R → R
(gof) (x) = g (f (x))

= g (x+1)
= sin (x+1)

(vii) Given f (x) = x+1, g (x) = 2x + 3
f: R→R ; g: R → R
Now we have to compute fog
Clearly, the range of g is a subset of the domain of f.
⇒ fog: R→ R
(fog) (x) = f (g (x))
= f (2x+3)
= 2x + 3 + 1
= 2x + 4
Now we have to compute gof
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R → R
(gof) (x) = g (f (x))
= g (x+1)
= 2 (x + 1) + 3
= 2x + 5

(viii) Given f (x) = c, g (x) = sin x^2
f: R → {c} ; g: R→ [0, 1]
Now we have to compute fog
Clearly, the range of g is a subset of the domain of f.
fog: R→R
(fog) (x) = f (g (x))
= f (sin x^2)
= c
Now we have to compute gof,
Clearly, the range of f is a subset of the domain of g.
⇒ fog: R→ R
(gof) (x) = g (f (x))
= g (c)
= sin c^2

(ix) Given f (x) = x^2+ 2 and g (x) = 1 - 1 / (1 - x)
f: R → [2, ∞)

For domain of g: $1 - x \neq 0$

$\Rightarrow x \neq 1$

$\Rightarrow$ Domain of g = R – {1}

g (x)= 1 – [1/(1 - x)] = (1 – x - 1)/ (1 - x) = -x/(1 - x)

For range of g

y = (- x)/ (1 - x)

$\Rightarrow y - xy = -x$

$\Rightarrow y = xy - x$

$\Rightarrow y = x(y-1)$

$\Rightarrow x = y/(y - 1)$

Range of g = R – {1}

So, g: R – {1} → R – {1}

Now we have to compute fog

Clearly, the range of g is a subset of the domain of f.

$\Rightarrow$ fog: R – {1} → R

(fog) (x) = f (g (x))

= f (-x/ (1 - x))

$= ((-x)/(1 - x))^2 + 2$

$= (x^2 + 2x^2 + 2 - 4x) / (1 - x)^2$

$= (3x^2 - 4x + 2)/ (1 - x)^2$

Now we have to compute gof

Clearly, the range of f is a subset of the domain of g.

$\Rightarrow$ gof: R→R

(gof) (x) = g (f (x))

$= g(x^2 + 2)$

$= 1 - 1 / (1 - (x^2 + 2))$

$= - 1/ (1 - (x^2 + 2))$

$= (x^2 + 2)/ (x^2 + 1)$

2. Let $f(x) = x^2 + x + 1$ and $g(x) = \sin x$. Show that fog ≠ gof.

Solution:

Given $f(x) = x^2 + x + 1$ and $g(x) = \sin x$

Now we have to prove fog ≠ gof

(fog) (x) = f (g (x))

= f (sin x)

$= \sin^2 x + \sin x + 1$

And (gof) (x) = g (f (x))
= g (x^2+ x + 1)
= sin (x^2+ x + 1)
So, fog ≠ gof.

3. If f(x) = |x|, prove that fof = f.

Solution:
Given f(x) = |x|,
Now we have to prove that fof = f.
Consider (fof) (x) = f (f (x))
= f (|x|)
= ||x||
= |x|
= f (x)
So,
(fof) (x) = f (x), ∀x ∈ R
Hence, fof = f

4. If f(x) = 2x + 5 and g(x) = x^2 + 1 be two real functions, then describe each of the following functions:
(i) fog
(ii) gof
(iii) fof
(iv) f^2
Also, show that fof ≠ f^2

Solution:
f(x) and g(x) are polynomials.
⇒ f: R → R and g: R → R.
So, fog: R → R and gof: R → R.
(i) (fog) (x) = f (g (x))
= f (x^2 + 1)
= 2 (x^2 + 1) + 5
=$2x^2$ + 2 + 5
= $2x^2$ +7

(ii) (gof) (x) = g (f (x))
= g (2x +5)
= $(2x + 5)^2 + 1$
= $4x^2 + 20x + 26$

(iii) (fof) (x) = f (f (x))
= f (2x +5)
= 2 (2x + 5) + 5
= 4x + 10 + 5
= 4x + 15

(iv) f^2 (x) = f (x) x f (x)
= (2x + 5) (2x + 5)
= $(2x + 5)^2$
= $4x^2 + 20x + 25$
Hence, from (iii) and (iv) clearly fof ≠ f^2

5. If f(x) = sin x and g(x) = 2x be two real functions, then describe gof and fog. Are these equal functions?

Solution:
Given f(x) = sin x and g(x) = 2x
We know that
f: R→ [−1, 1] and g: R→ R
Clearly, the range of f is a subset of the domain of g.
gof: R→ R
(gof) (x) = g (f (x))
= g (sin x)
= 2 sin x
Clearly, the range of g is a subset of the domain of f.
fog: R → R
So, (fog) (x) = f (g (x))
= f (2x)
= sin (2x)
Clearly, fog ≠ gof
Hence they are not equal functions.

6. Let f, g, h be real functions given by f(x) = sin x, g (x) = 2x and h (x) = cos x. Prove that fog = go (f h).

Solution:

Given that f(x) = sin x, g (x) = 2x and h (x) = cos x

We know that f: R→ [−1, 1] and g: R→ R

Clearly, the range of g is a subset of the domain of f.

fog: R → R

Now, (f h) (x) = f (x) h (x) = (sin x) (cos x) = ½ sin (2x)

Domain of f h is R.

Since range of sin x is [-1, 1], −1 ≤ sin 2x ≤ 1

⇒ -1/2 ≤ sin x/2 ≤ 1/2

Range of f h = [-1/2, 1/2]

So, (f h): R → [(-1)/2, 1/2]

Clearly, range of f h is a subset of g.

⇒ go (f h): R → R

⇒ Domains of fog and go (f h) are the same.

So, (fog) (x) = f (g (x))

= f (2x)

= sin (2x)

And (go (f h)) (x) = g ((f(x). h(x))

= g (sin x cos x)

= 2sin x cos x

= sin (2x)

⇒ (fog) (x) = (go (f h)) (x), ∀x ∈ R

Hence, fog = go (f h)