EXERCISE 3.4

Find f + g, f – g, cf (c ∈ R, c ≠ 0), fg, 1/f and f/g in each of the following:
(i) f (x) = x^3 + 1 and g (x) = x + 1
(ii) f (x) = √(x-1) and g (x) = √(x+1)
Solution:
(i) f (x) = $x^3 + 1$ and g(x) = x + 1
We have f(x): R → R and g(x): R → R

(a) f + g
We know, (f + g) (x) = f(x) + g(x)
(f + g) (x) = $x^3 + 1 + x + 1$
= $x^3 + x + 2$
So, (f + g) (x): R → R
∴ f + g: R → R is given by (f + g) (x) = $x^3 + x + 2$

(b) f – g
We know, (f – g) (x) = f(x) – g(x)
(f – g) (x) = $x^3 + 1 – (x + 1)$
= $x^3 + 1 – x – 1$
= $x^3 – x$
So, (f – g) (x): R → R
∴ f – g: R → R is given by (f – g) (x) = $x^3 – x$

(c) cf (c ∈ R, c ≠ 0)
We know, (cf) (x) = c × f(x)
(cf)(x) = $c(x^3 + 1)$
= $cx^3 + c$
So, (cf) (x) : R → R
∴ cf: R → R is given by (cf) (x) = $cx^3 + c$

(d) fg
We know, (fg) (x) = f(x) g(x)
(fg) (x) = $(x^3 + 1) (x + 1)$
= $(x + 1) (x^2 – x + 1) (x + 1)$
= $(x + 1)^2 (x^2 – x + 1)$
So, (fg) (x): R → R
∴ fg: R → R is given by (fg) (x) = $(x + 1)^2(x^2 – x + 1)$

(e) 1/f

We know, (1/f) (x) = 1/f (x)

$1/f(x) = 1 / (x^3 + 1)$

Observe that 1/f(x) is undefined when f(x) = 0 or when x = – 1.

So, 1/f: R – {–1} → R is given by $1/f(x) = 1 / (x^3 + 1)$

(f) f/g

We know, (f/g) (x) = f(x)/g(x)

$(f/g)(x) = (x^3 + 1) / (x + 1)$

Observe that $(x^3 + 1) / (x + 1)$ is undefined when g(x) = 0 or when x = –1.

Using $x^3 + 1 = (x + 1)(x^2 – x + 1)$, we have

$(f/g)(x) = [(x+1)(x^2 - x+1)/(x+1)]$

$= x^2 – x + 1$

∴ f/g: R – {–1} → R is given by $(f/g)(x) = x^2 – x + 1$

(ii) f (x) = √(x-1) and g (x) = √(x+1)

We have f(x): [1, ∞) → R^+ and g(x): [–1, ∞) → R^+ as real square root is defined only for non-negative numbers.

(a) f + g

We know, (f + g) (x) = f(x) + g(x)

(f+g) (x) = √(x-1) + √(x+1)

Domain of (f + g) = Domain of f ∩ Domain of g

Domain of (f + g) = [1, ∞) ∩ [–1, ∞)

Domain of (f + g) = [1, ∞)

∴ f + g: [1, ∞) → R is given by (f+g) (x) = √(x-1) + √(x+1)

(b) f – g

We know, (f – g) (x) = f(x) – g(x)

(f-g) (x) = √(x-1) - √(x+1)

Domain of (f – g) = Domain of f ∩ Domain of g

Domain of (f – g) = [1, ∞) ∩ [–1, ∞)

Domain of (f – g) = [1, ∞)

∴ f – g: [1, ∞) → R is given by (f-g) (x) = √(x-1) - √(x+1)

(c) cf (c ∈ R, c ≠ 0)

We know, (cf) (x) = c × f(x)

(cf) (x) = c√(x-1)

Domain of (cf) = Domain of f

Domain of (cf) = [1, ∞)
∴ cf: [1, ∞) → R is given by (cf) (x) = c√(x-1)

(d) fg
We know, (fg) (x) = f(x) g(x)
(fg) (x) = √(x-1) √(x+1)
= √(x^2 -1)
Domain of (fg) = Domain of f ∩ Domain of g
Domain of (fg) = [1, ∞) ∩ [–1, ∞)
Domain of (fg) = [1, ∞)
∴ fg: [1, ∞) → R is given by (fg) (x) = √(x^2 -1)

(e) 1/f
We know, (1/f) (x) = 1/f(x)
(1/f) (x) = 1/√(x-1)
Domain of (1/f) = Domain of f
Domain of (1/f) = [1, ∞)
Observe that 1/√(x-1) is also undefined when x – 1 = 0 or x = 1.
∴ 1/f: (1, ∞) → R is given by (1/f) (x) = 1/√(x-1)

(f) f/g
We know, (f/g) (x) = f(x)/g(x)
(f/g) (x) = √(x-1)/√(x+1)
(f/g) (x) = √[(x-1)/(x+1)]
Domain of (f/g) = Domain of f ∩ Domain of g
Domain of (f/g) = [1, ∞) ∩ [–1, ∞)
Domain of (f/g) = [1, ∞)
∴ f/g: [1, ∞) → R is given by (f/g) (x) = √[(x-1)/(x+1)]

2. Let f(x) = 2x + 5 and g(x) = x^2 + x. Describe
(i) f + g
(ii) f – g
(iii) fg
(iv) f/g
Find the domain in each case.
Solution:
Given:
f(x) = 2x + 5 and g(x) = x^2 + x
Both f(x) and g(x) are defined for all x ∈ R.

So, domain of f = domain of g = R

(i) f + g

We know, (f + g)(x) = f(x) + g(x)

(f + g)(x) = $2x + 5 + x^2 + x$

$= x^2 + 3x + 5$

(f + g)(x) Is defined for all real numbers x.

∴ The domain of (f + g) is R

(ii) f – g

We know, (f – g)(x) = f(x) – g(x)

(f – g)(x) = $2x + 5 - (x^2 + x)$

$= 2x + 5 - x^2 - x$

$= 5 + x - x^2$

(f – g)(x) is defined for all real numbers x.

∴ The domain of (f – g) is R

(iii) fg

We know, (fg)(x) = f(x)g(x)

(fg)(x) = $(2x + 5)(x^2 + x)$

$= 2x(x^2 + x) + 5(x^2 + x)$

$= 2x^3 + 2x^2 + 5x^2 + 5x$

$= 2x^3 + 7x^2 + 5x$

(fg)(x) is defined for all real numbers x.

∴ The domain of fg is R

(iv) f/g

We know, (f/g) (x) = f(x)/g(x)

(f/g) (x) = $(2x+5)/(x^2+x)$

(f/g) (x) is defined for all real values of x, except for the case when $x^2 + x = 0$.

$x^2 + x = 0$

$x(x + 1) = 0$

$x = 0$ or $x + 1 = 0$

$x = 0$ or -1

When x = 0 or –1, (f/g) (x) will be undefined as the division result will be indeterminate.

∴ The domain of f/g = R – {–1, 0}

3. If f(x) be defined on [–2, 2] and is given by $f(x) = \begin{cases} -1, -2 \leq x \leq 0 \\ x - 1, 0 < x \leq 2 \end{cases}$ **and g(x) = f(|x|) + |f(x)|. Find g(x).**

Solution:

Given:

$$f(x)=\begin{cases}-1, -2\le x\le 0\\ x-1, 0<x\le 2\end{cases}$$ and

g(x) = f(|x|) + |f(x)|

Now we have,

$$f(|x|)=\begin{cases}-1, -2\le |x|\le 0\\ |x|-1, 0<|x|\le 2\end{cases}$$

However, |x| ≥ 0 ⇒ f(|x|) = |x| – 1 when 0 < |x| ≤ 2

We also have,

$$|f(x)|=\begin{cases}|-1|, -2\le x\le 0\\ |x-1|, 0<x\le 2\end{cases}$$

$$=\begin{cases}1, -2\le x\le 0\\ |x-1|, 0<x\le 2\end{cases}$$

We also know,

$$|x-1|=\begin{cases}-(x-1), x-1<0\\ x-1, x-1\ge 0\end{cases}$$

$$=\begin{cases}-(x-1), x<1\\ x-1, x\ge 1\end{cases}$$

Here, we shall only the range between [0, 2].

$$|x-1|=\begin{cases}-(x-1), 0<x<1\\ x-1, 1\le x\le 2\end{cases}$$

Substituting this value of |x – 1| in |f(x)|, we get

$$|f(x)|=\begin{cases}1, -2\le x\le 0\\ -(x-1), 0<x<1\\ x-1, 1\le x\le 2\end{cases}$$

$$=\begin{cases}1, -2\le x\le 0\\ 1-x, 0<x<1\\ x-1, 1\le x\le 2\end{cases}$$

Now, we need to find g(x)

g(x) = f(|x|) + |f(x)|

$$= |x|-1 \text{ when } 0<|x|\le 2 + \begin{cases}1, -2\le x\le 0\\ 1-x, 0<x<1\\ x-1, 1\le x\le 2\end{cases}$$

$$g(x)=\begin{cases}-x-1, -2\le x\le 0\\ x-1, 0<x<1\\ x-1, 1\le x\le 2\end{cases}+\begin{cases}1, -2\le x\le 0\\ 1-x, 0<x<1\\ x-1, 1\le x\le 2\end{cases}$$
$$=\begin{cases}-x-1+1, -2\le x\le 0\\ x-1+1-x, 0<x<1\\ x-1+x-1, 1\le x\le 2\end{cases}$$
$$=\begin{cases}-x, -2\le x\le 0\\ 0, 0<x<1\\ 2(x-1), 1\le x\le 2\end{cases}$$

∴ g(x) = f(|x|) + |f(x)| |
$$=\begin{cases}-x, -2\le x\le 0\\ 0, 0<x<1\\ 2(x-1), 1\le x\le 2\end{cases}$$

4. Let f, g be two real functions defined by f(x) = √(x+1) and g(x) = √(9-x^2). Then, describe each of the following functions.
(i) f + g
(ii) g – f
(iii) fg
(iv) f/g
(v) g/f
(vi) 2f - √5g
(vii) f^2 + 7f
(viii) 5/g
Solution:
Given:
f(x) = √(x+1) and g(x) = √(9-x^2)
We know the square of a real number is never negative.
So, f(x) takes real values only when x + 1 ≥ 0
x ≥ –1, x ∈ [–1, ∞)
Domain of f = [–1, ∞)

Similarly, g(x) takes real values only when 9 – x^2 ≥ 0
9 ≥ x^2
x^2 ≤ 9
x^2 – 9 ≤ 0
x^2 – 3^2 ≤ 0
(x + 3)(x – 3) ≤ 0

$x \geq -3$ and $x \leq 3$
$\therefore x \in [-3, 3]$
Domain of g = $[-3, 3]$

(i) f + g
We know, $(f + g)(x) = f(x) + g(x)$
$(f + g)(x) = \sqrt{(x+1)} + \sqrt{(9-x^2)}$
Domain of f + g = Domain of f ∩ Domain of g
= $[-1, \infty) \cap [-3, 3]$
= $[-1, 3]$
$\therefore$ f + g: $[-1, 3] \rightarrow R$ is given by $(f + g)(x) = f(x) + g(x) = \sqrt{(x+1)} + \sqrt{(9-x^2)}$

(ii) g – f
We know, $(g - f)(x) = g(x) - f(x)$
$(g - f)(x) = \sqrt{(9-x^2)} - \sqrt{(x+1)}$
Domain of g – f = Domain of g ∩ Domain of f
= $[-3, 3] \cap [-1, \infty)$
= $[-1, 3]$
$\therefore$ g – f: $[-1, 3] \rightarrow R$ is given by $(g - f)(x) = g(x) - f(x) = \sqrt{(9-x^2)} - \sqrt{(x+1)}$

(iii) fg
We know, $(fg)(x) = f(x)g(x)$
$(fg)(x) = \sqrt{(x+1)}\ \sqrt{(9-x^2)}$
$= \sqrt{[(x+1)(9-x^2)]}$
$= \sqrt{[x(9-x^2) + (9-x^2)]}$
$= \sqrt{(9x-x^3+9-x^2)}$
$= \sqrt{(9+9x-x^2-x^3)}$
Domain of fg = Domain of f ∩ Domain of g
= $[-1, \infty) \cap [-3, 3]$
= $[-1, 3]$
$\therefore$ fg: $[-1, 3] \rightarrow R$ is given by $(fg)(x) = f(x)\,g(x) = \sqrt{(x+1)}\ \sqrt{(9-x^2)} = \sqrt{(9+9x-x^2-x^3)}$

(iv) f/g
We know, $(f/g)(x) = f(x)/g(x)$
$(f/g)(x) = \sqrt{(x+1)} / \sqrt{(9-x^2)}$
$= \sqrt{[(x+1) / (9-x^2)]}$
Domain of f/g = Domain of f ∩ Domain of g
= $[-1, \infty) \cap [-3, 3]$
= $[-1, 3]$

However, (f/g) (x) is defined for all real values of $x \in [-1, 3]$, except for the case when $9 - x^2 = 0$ or $x = \pm 3$

When $x = \pm 3$, (f/g) (x) will be undefined as the division result will be indeterminate.

Domain of f/g = [–1, 3] – {–3, 3}

Domain of f/g = [–1, 3)

∴ f/g: [–1, 3) → R is given by $(f/g)(x) = f(x)/g(x) = \sqrt{(x+1)} / \sqrt{(9-x^2)}$

(v) g/f

We know, (g/f) (x) = g(x)/f(x)

$(g/f)(x) = \sqrt{(9-x^2)} / \sqrt{(x+1)}$

$= \sqrt{[(9-x^2) / (x+1)]}$

Domain of g/f = Domain of f ∩ Domain of g

= [–1, ∞) ∩ [–3, 3]

= [–1, 3]

However, (g/f) (x) is defined for all real values of $x \in [-1, 3]$, except for the case when $x + 1 = 0$ or $x = -1$

When $x = -1$, (g/f) (x) will be undefined as the division result will be indeterminate.

Domain of g/f = [–1, 3] – {–1}

Domain of g/f = (–1, 3]

∴ g/f: (–1, 3] → R is given by $(g/f)(x) = g(x)/f(x) = \sqrt{(9-x^2)} / \sqrt{(x+1)}$

(vi) $2f - \sqrt{5}g$

We know, $(2f - \sqrt{5}g)(x) = 2f(x) - \sqrt{5}g(x)$

$(2f - \sqrt{5}g)(x) = 2f(x) - \sqrt{5}g(x)$

$= 2\sqrt{(x+1)} - \sqrt{5}\sqrt{(9-x^2)}$

$= 2\sqrt{(x+1)} - \sqrt{(45 - 5x^2)}$

Domain of $2f - \sqrt{5}g$ = Domain of f ∩ Domain of g

= [–1, ∞) ∩ [–3, 3]

= [–1, 3]

∴ $2f - \sqrt{5}g$: [–1, 3] → R is given by $(2f - \sqrt{5}g)(x) = 2f(x) - \sqrt{5}g(x) = 2\sqrt{(x+1)} - \sqrt{(45 - 5x^2)}$

(vii) $f^2 + 7f$

We know, $(f^2 + 7f)(x) = f^2(x) + (7f)(x)$

$(f^2 + 7f)(x) = f(x)\,f(x) + 7f(x)$

$= \sqrt{(x+1)}\,\sqrt{(x+1)} + 7\sqrt{(x+1)}$

$= x + 1 + 7\sqrt{(x+1)}$

Domain of $f^2 + 7f$ is same as domain of f.

Domain of $f^2 + 7f$ = [–1, ∞)

∴ $f^2 + 7f$: [–1, ∞) → R is given by $(f^2 + 7f)(x) = f(x)\,f(x) + 7f(x) = x + 1 + 7\sqrt{(x+1)}$

(viii) 5/g
We know, (5/g) (x) = 5/g(x)
(5/g) (x) = $5/\sqrt{(9-x^2)}$
Domain of 5/g = Domain of g = [–3, 3]
However, (5/g) (x) is defined for all real values of x ∈ [–3, 3], except for the case when $9 - x^2 = 0$ or x = ± 3
When x = ±3, (5/g) (x) will be undefined as the division result will be indeterminate.
Domain of 5/g = [–3, 3] – {–3, 3}
= (–3, 3)
∴ 5/g: (–3, 3) → R is given by (5/g) (x) = 5/g(x) = $5/\sqrt{(9-x^2)}$

5. If f(x) = $\log_e$ (1 – x) and g(x) = [x], then determine each of the following functions:
(i) f + g
(ii) fg
(iii) f/g
(iv) g/f
Also, find (f + g) (–1), (fg) (0), (f/g) (1/2) and (g/f) (1/2).
Solution:
Given:
f(x) = $\log_e$ (1 – x) and g(x) = [x]
We know, f(x) takes real values only when 1 – x > 0
1 > x
x < 1, ∴ x ∈ (–∞, 1)
Domain of f = (–∞, 1)

Similarly, g(x) is defined for all real numbers x.
Domain of g = [x], x ∈ R
= R

(i) f + g
We know, (f + g) (x) = f(x) + g(x)
(f + g) (x) = $\log_e$ (1 – x) + [x]
Domain of f + g = Domain of f ∩ Domain of g
Domain of f + g = (–∞, 1) ∩ R
= (–∞, 1)
∴ f + g: (–∞, 1) → R is given by (f + g) (x) = $\log_e$ (1 – x) + [x]

(ii) fg

We know, (fg) (x) = f(x) g(x)
(fg) (x) = $\log_e (1 - x) \times [x]$
= $[x] \log_e (1 - x)$
Domain of fg = Domain of f ∩ Domain of g
= $(-\infty, 1) \cap R$
= $(-\infty, 1)$
∴ fg: $(-\infty, 1) \to R$ is given by (fg) (x) = $[x] \log_e (1 - x)$

(iii) f/g
We know, (f/g) (x) = f(x)/g(x)
(f/g) (x) = $\log_e (1 - x) / [x]$
Domain of f/g = Domain of f ∩ Domain of g
= $(-\infty, 1) \cap R$
= $(-\infty, 1)$
However, (f/g) (x) is defined for all real values of $x \in (-\infty, 1)$, except for the case when $[x] = 0$.
We have, $[x] = 0$ when $0 \le x < 1$ or $x \in [0, 1)$
When $0 \le x < 1$, (f/g) (x) will be undefined as the division result will be indeterminate.
Domain of f/g = $(-\infty, 1) - [0, 1)$
= $(-\infty, 0)$
∴ f/g: $(-\infty, 0) \to R$ is given by (f/g) (x) = $\log_e (1 - x) / [x]$

(iv) g/f
We know, (g/f) (x) = g(x)/f(x)
(g/f) (x) = $[x] / \log_e (1 - x)$
However, (g/f) (x) is defined for all real values of $x \in (-\infty, 1)$, except for the case when $\log_e (1 - x) = 0$.
$\log_e (1 - x) = 0 \Rightarrow 1 - x = 1$ or $x = 0$
When x = 0, (g/f) (x) will be undefined as the division result will be indeterminate.
Domain of g/f = $(-\infty, 1) - \{0\}$
= $(-\infty, 0) \cup (0, 1)$
∴ g/f: $(-\infty, 0) \cup (0, 1) \to R$ is given by (g/f) (x) = $[x] / \log_e (1 - x)$

(a) We need to find (f + g) (–1).
We have, (f + g) (x) = $\log_e (1 - x) + [x]$, $x \in (-\infty, 1)$
Substituting x = –1 in the above equation, we get
(f + g)(–1) = $\log_e (1 - (-1)) + [-1]$
= $\log_e (1 + 1) + (-1)$
= $\log_e 2 - 1$

∴ $(f + g)(-1) = \log_e 2 - 1$

(b) We need to find (fg) (0).
We have, $(fg)(x) = [x] \log_e (1 - x)$, $x \in (-\infty, 1)$
Substituting x = 0 in the above equation, we get
$(fg)(0) = [0] \log_e (1 - 0)$
$= 0 \times \log_e 1$
∴ $(fg)(0) = 0$

(c) We need to find (f/g) (1/2)
We have, $(f/g)(x) = \log_e (1 - x) / [x]$, $x \in (-\infty, 0)$
However, 1/2 is not in the domain of f/g.
∴ (f/g) (1/2) does not exist.

(d) We need to find (g/f) (1/2)
We have, $(g/f)(x) = [x] / \log_e (1 - x)$, $x \in (-\infty, 0) \cup (0, \infty)$
Substituting x=1/2 in the above equation, we get
$(g/f)(1/2) = [x] / \log_e (1 - x)$
$= (1/2)/ \log_e (1 - 1/2)$
$= 0.5/ \log_e (1/2)$
$= 0 / \log_e (1/2)$
$= 0$
∴ $(g/f)(1/2) = 0$