EXERCISE 3.2

1. Let '*' be a binary operation on N defined by a * b = l.c.m. (a, b) for all a, b ∈ N
(i) Find 2 * 4, 3 * 5, 1 * 6.
(ii) Check the commutativity and associativity of '*' on N.

Solution:
(i) Given a * b = 1.c.m. (a, b)
2 * 4 = l.c.m. (2, 4)
= 4
3 * 5 = l.c.m. (3, 5)
= 15
1 * 6 = l.c.m. (1, 6)
= 6

(ii) We have to prove commutativity of *
Let a, b ∈ N
a * b = l.c.m (a, b)
= l.c.m (b, a)
= b * a
Therefore
a * b = b * a ∀ a, b ∈ N
Thus * is commutative on N.
Now we have to prove associativity of *
Let a, b, c ∈ N
a * (b * c) = a * l.c.m. (b, c)
= l.c.m. (a, (b, c))
= l.c.m (a, b, c)
(a * b) * c = l.c.m. (a, b) * c
= l.c.m. ((a, b), c)
= l.c.m. (a, b, c)
Therefore
(a * (b * c) = (a * b) * c, ∀ a, b , c ∈ N
Thus, * is associative on N.

2. Determine which of the following binary operation is associative and which is

commutative:

(i) * on N defined by a * b = 1 for all a, b ∈ N

(ii) * on Q defined by a * b = (a + b)/2 for all a, b ∈ Q

Solution:

(i) We have to prove commutativity of *
Let a, b ∈ N
a * b = 1
b * a = 1
Therefore,
a * b = b * a, for all a, b ∈ N
Thus * is commutative on N.
Now we have to prove associativity of *
Let a, b, c ∈ N
Then a * (b * c) = a * (1)
= 1
(a * b) *c = (1) * c
= 1
Therefore a * (b * c) = (a * b) *c for all a, b, c ∈ N
Thus, * is associative on N.

(ii) First we have to prove commutativity of *
Let a, b ∈ N
a * b = (a + b)/2
= (b + a)/2
= b * a
Therefore, a * b = b * a, ∀ a, b ∈ N
Thus * is commutative on N.
Now we have to prove associativity of *
Let a, b, c ∈ N
a * (b * c) = a * (b + c)/2
= [a + (b + c)]/2
= (2a + b + c)/4
Now, (a * b) * c = (a + b)/2 * c
= [(a + b)/2 + c] /2
= (a + b + 2c)/4
Thus, a * (b * c) ≠ (a * b) * c

If a = 1, b= 2, c = 3
1 * (2 * 3) = 1 * (2 + 3)/2
= 1 * (5/2)
= [1 + (5/2)]/2
= 7/4
(1 * 2) * 3 = (1 + 2)/2 * 3
= 3/2 * 3
= [(3/2) + 3]/2
= 4/9
Therefore, there exist a = 1, b = 2, c = 3 ∈ N such that a * (b * c) ≠ (a * b) * c
Thus, * is not associative on N.

3. Let A be any set containing more than one element. Let '*' be a binary operation on A defined by a * b = b for all a, b ∈ A Is '*' commutative or associative on A?

Solution:
Let a, b ∈ A
Then, a * b = b
b * a = a
Therefore a * b ≠ b * a
Thus, * is not commutative on A
Now we have to check associativity:
Let a, b, c ∈ A
a * (b * c) = a * c
= c
Therefore
a * (b * c) = (a * b) * c, ∀ a, b, c ∈ A
Thus, * is associative on A

4. Check the commutativity and associativity of each of the following binary operations:
(i) '*' on Z defined by a * b = a + b + a b for all a, b ∈ Z
(ii) '*' on N defined by a * b = 2^{ab} for all a, b ∈ N
(iii) '*' on Q defined by a * b = a – b for all a, b ∈ Q
(iv) '⊙' on Q defined by a ⊙ b = $a^2 + b^2$ for all a, b ∈ Q
(v) 'o' on Q defined by a o b = (ab/2) for all a, b ∈ Q
(vi) '*' on Q defined by a * b = ab^2 for all a, b ∈ Q

(vii) '*' on Q defined by a * b = a + a b for all a, b ∈ Q
(viii) '*' on R defined by a * b = a + b -7 for all a, b ∈ R
(ix) '*' on Q defined by a * b = $(a – b)^2$ for all a, b ∈ Q
(x) '*' on Q defined by a * b = a b + 1 for all a, b ∈ Q
(xi) '*' on N defined by a * b = a^b for all a, b ∈ N
(xii) '*' on Z defined by a * b = a – b for all a, b ∈ Z
(xiii) '*' on Q defined by a * b = (ab/4) for all a, b ∈ Q
(xiv) '*' on Z defined by a * b = a + b – ab for all a, b ∈ Z
(xv) '*' on Q defined by a * b = gcd (a, b) for all a, b ∈ Q

Solution:
(i) First we have to check commutativity of *
Let a, b ∈ Z
Then a * b = a + b + ab
= b + a + ba
= b * a
Therefore,
a * b = b * a, ∀ a, b ∈ Z
Now we have to prove associativity of *
Let a, b, c ∈ Z, Then,
a * (b * c) = a * (b + c + b c)
= a + (b + c + b c) + a (b + c + b c)
= a + b + c + b c + a b + a c + a b c
(a * b) * c = (a + b + a b) * c
= a + b + a b + c + (a + b + a b) c
= a + b + a b + c + a c + b c + a b c
Therefore,
a * (b * c) = (a * b) * c, ∀ a, b, c ∈ Z
Thus, * is associative on Z.

(ii) First we have to check commutativity of *
Let a, b ∈ N
a * b = 2^{ab}
= 2^{ba}
= b * a
Therefore, a * b = b * a, ∀ a, b ∈ N
Thus, * is commutative on N

Now we have to check associativity of *
Let a, b, c ∈ N
Then, a * (b * c) = a * (2^{bc})
= $2^{a*2^{bc}}$
(a * b) * c = (2^{ab}) * c
= $2^{ab*2^{c}}$
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on N

(iii) First we have to check commutativity of *
Let a, b ∈ Q, then
a * b = a – b
b * a = b – a
Therefore, a * b ≠ b * a
Thus, * is not commutative on Q
Now we have to check associativity of *
Let a, b, c ∈ Q, then
a * (b * c) = a * (b – c)
= a – (b – c)
= a – b + c
(a * b) * c = (a – b) * c
= a – b – c
Therefore,
a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Q

(iv) First we have to check commutativity of ⊙
Let a, b ∈ Q, then
a ⊙ b = $a^2 + b^2$
= $b^2 + a^2$
= b ⊙ a
Therefore, a ⊙ b = b ⊙ a, ∀ a, b ∈ Q
Thus, ⊙ on Q
Now we have to check associativity of ⊙
Let a, b, c ∈ Q, then
a ⊙ (b ⊙ c) = a ⊙ ($b^2 + c^2$)
= $a^2 + (b^2 + c^2)^2$

$= a^2 + b^4 + c^4 + 2b^2c^2$
$(a \odot b) \odot c = (a^2 + b^2) \odot c$
$= (a^2 + b^2)^2 + c^2$
$= a^4 + b^4 + 2a^2b^2 + c^2$
Therefore,
$(a \odot b) \odot c \neq a \odot (b \odot c)$
Thus, $\odot$ is not associative on Q.

(v) First we have to check commutativity of o
Let a, b ∈ Q, then
a o b = (ab/2)
= (b a/2)
= b o a
Therefore, a o b = b o a, ∀ a, b ∈ Q
Thus, o is commutative on Q
Now we have to check associativity of o
Let a, b, c ∈ Q, then
a o (b o c) = a o (b c/2)
= [a (b c/2)]/2
= [a (b c/2)]/2
= (a b c)/4
(a o b) o c = (ab/2) o c
= [(ab/2) c] /2
= (a b c)/4
Therefore a o (b o c) = (a o b) o c, ∀ a, b, c ∈ Q
Thus, o is associative on Q.

(vi) First we have to check commutativity of *
Let a, b ∈ Q, then
$a * b = ab^2$
$b * a = ba^2$
Therefore,
$a * b \neq b * a$
Thus, * is not commutative on Q
Now we have to check associativity of *
Let a, b, c ∈ Q, then
$a * (b * c) = a * (bc^2)$

$= a (bc^2)^2$
$= ab^2 c^4$
$(a * b) * c = (ab^2) * c$
$= ab^2c^2$
Therefore $a * (b * c) \neq (a * b) * c$
Thus, * is not associative on Q.

(vii) First we have to check commutativity of *
Let a, b ∈ Q, then
a * b = a + ab
b * a = b + ba
= b + ab
Therefore, a * b ≠ b * a
Thus, * is not commutative on Q.
Now we have to prove associativity on Q.
Let a, b, c ∈ Q, then
a * (b * c) = a * (b + b c)
= a + a (b + b c)
= a + ab + a b c
(a * b) * c = (a + a b) * c
= (a + a b) + (a + a b) c
= a + a b + a c + a b c
Therefore a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Q.

(viii) First we have to check commutativity of *
Let a, b ∈ R, then
a * b = a + b – 7
= b + a – 7
= b * a
Therefore,
a * b = b * a, for all a, b ∈ R
Thus, * is commutative on R
Now we have to prove associativity of * on R.
Let a, b, c ∈ R, then
a * (b * c) = a * (b + c – 7)
= a + b + c -7 -7

= a + b + c – 14
(a * b) * c = (a + b – 7) * c
= a + b – 7 + c - 7
= a + b + c – 14
Therefore,
a * (b * c) = (a * b) * c, for all a, b, c ∈ R
Thus, * is associative on R.

(ix) First we have to check commutativity of *
Let a, b ∈ Q, then
a * b = $(a – b)^2$
= $(b – a)^2$
= b * a
Therefore,
a * b = b * a, for all a, b ∈ Q
Thus, * is commutative on Q
Now we have to prove associativity of * on Q
Let a, b, c ∈ Q, then
a * (b * c) = a * $(b – c)^2$
= a * $(b^2 + c^2 – 2\,b\,c)$
= $(a – b^2 – c^2 + 2bc)^2$
(a * b) * c = $(a – b)^2$ * c
= $(a^2 + b^2 – 2ab)$ * c
= $(a^2 + b^2 – 2ab – c)^2$
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Q.

(x) First we have to check commutativity of *
Let a, b ∈ Q, then
a * b = ab + 1
= ba + 1
= b * a
Therefore
a * b = b * a, for all a, b ∈ Q
Thus, * is commutative on Q
Now we have to prove associativity of * on Q
Let a, b, c ∈ Q, then

a * (b * c) = a * (bc + 1)
= a (b c + 1) + 1
= a b c + a + 1
(a * b) * c = (ab + 1) * c
= (ab + 1) c + 1
= a b c + c + 1
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Q.

(xi) First we have to check commutativity of *
Let a, b ∈ N, then
a * b = a^b
b * a = b^a
Therefore, a * b ≠ b * a
Thus, * is not commutative on N.
Now we have to check associativity of *
a * (b * c) = a * (b^c)
= a^{b^c}
(a * b) * c = (a^b) * c
= $(a^b)^c$
= a^{bc}
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on N

(xii) First we have to check commutativity of *
Let a, b ∈ Z, then
a * b = a – b
b * a = b – a
Therefore,
a * b ≠ b * a
Thus, * is not commutative on Z.
Now we have to check associativity of *
Let a, b, c ∈ Z, then
a * (b * c) = a * (b – c)
= a – (b – c)
= a – (b + c)
(a * b) * c = (a – b) – c

= a – b – c
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Z

(xiii) First we have to check commutativity of *
Let a, b ∈ Q, then
a * b = (ab/4)
= (ba/4)
= b * a
Therefore, a * b = b * a, for all a, b ∈ Q
Thus, * is commutative on Q
Now we have to check associativity of *
Let a, b, c ∈ Q, then
a * (b * c) = a * (b c/4)
= [a (b c/4)]/4
= (a b c/16)
(a * b) * c = (ab/4) * c
= [(ab/4) c]/4
= a b c/16
Therefore,
a * (b * c) = (a * b) * c for all a, b, c ∈ Q
Thus, * is associative on Q.

(xiv) First we have to check commutativity of *
Let a, b ∈ Z, then
a * b = a + b – ab
= b + a – ba
= b * a
Therefore, a * b = b * a, for all a, b ∈ Z
Thus, * is commutative on Z.
Now we have to check associativity of *
Let a, b, c ∈ Z
a * (b * c) = a * (b + c – b c)
= a + b + c- b c – ab – ac + a b c
(a * b) * c = (a + b – a b) c
= a + b – ab + c – (a + b – ab) c
= a + b + c – ab – ac – bc + a b c

Therefore,
a * (b * c) = (a * b) * c, for all a, b, c ∈ Z
Thus, * is associative on Z.

(xv) First we have to check commutativity of *
Let a, b ∈ N, then
a * b = gcd (a, b)
= gcd (b, a)
= b * a
Therefore, a * b = b * a, for all a, b ∈ N
Thus, * is commutative on N.
Now we have to check associativity of *
Let a, b, c ∈ N
a * (b * c) = a * [gcd (a, b)]
= gcd (a, b, c)
(a * b) * c = [gcd (a, b)] * c
= gcd (a, b, c)
Therefore,
a * (b * c) = (a * b) * c, for all a, b, c ∈ N
Thus, * is associative on N.

5. If the binary operation o is defined by a0b = a + b – ab on the set Q – {-1} of all rational numbers other than 1, show that o is commutative on Q – [1].

Solution:
Let a, b ∈ Q – {-1}.
Then aob = a + b – ab
= b+ a – ba
= boa
Therefore,
aob = boa for all a, b ∈ Q – {-1}
Thus, o is commutative on Q – {-1}

6. Show that the binary operation * on Z defined by a * b = 3a + 7b is not commutative?

Solution:

Let a, b ∈ Z
a * b = 3a + 7b
b * a = 3b + 7a
Thus, a * b ≠ b * a
Let a = 1 and b = 2
1 * 2 = 3 × 1 + 7 × 2
= 3 + 14
= 17
2 * 1 = 3 × 2 + 7 × 1
= 6 + 7
= 13
Therefore, there exist a = 1, b = 2 ∈ Z such that a * b ≠ b * a
Thus, * is not commutative on Z.

7. On the set Z of integers a binary operation * is defined by a 8 b = ab + 1 for all a, b ∈ Z. Prove that * is not associative on Z.

Solution:
Let a, b, c ∈ Z
a * (b * c) = a * (bc + 1)
= a (bc + 1) + 1
= a b c + a + 1
(a * b) * c = (ab+ 1) * c
= (ab + 1) c + 1
= a b c + c + 1
Thus, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Z.