EXERCISE 13.4

Find the modulus and argument of the following complex numbers and hence express each of them in the polar form:

(i) $1 + i$

(ii) $\sqrt{3} + i$

(iii) $1 - i$

(iv) $(1 - i) / (1 + i)$

(v) $1/(1 + i)$

(vi) $(1 + 2i) / (1 – 3i)$

(vii) $\sin 120^o – i \cos 120^o$

(viii) $-16 / (1 + i\sqrt{3})$

Solution:

We know that the polar form of a complex number $Z = x + iy$ is given by $Z = |Z| (\cos \theta + i \sin \theta)$

Where,

$|Z|$ = modulus of complex number = $\sqrt{(x^2 + y^2)}$

$\theta = \arg(z)$ = argument of complex number = $\tan^{-1} (|y| / |x|)$

(i) $1 + i$

Given: $Z = 1 + i$

So now,

$|Z| = \sqrt{(x^2 + y^2)}$

$= \sqrt{(1^2 + 1^2)}$

$= \sqrt{(1 + 1)}$

$= \sqrt{2}$

$\theta = \tan^{-1} (|y| / |x|)$

$= \tan^{-1} (1 / 1)$

$= \tan^{-1} 1$

Since $x > 0$, $y > 0$ complex number lies in 1st quadrant and the value of θ is $0^0 \le \theta \le 90^0$.

$\theta = \pi/4$

$Z = \sqrt{2} (\cos (\pi/4) + i \sin (\pi/4))$

∴ Polar form of $(1 + i)$ is $\sqrt{2} (\cos (\pi/4) + i \sin (\pi/4))$

(ii) $\sqrt{3} + i$

Given: $Z = \sqrt{3} + i$

So now,

$|Z| = \sqrt{(x^2 + y^2)}$

$= \sqrt{((\sqrt{3})^2 + 1^2)}$

$= \sqrt{(3 + 1)}$
$= \sqrt{4}$
$= 2$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (1 / \sqrt{3})$
Since $x > 0$, $y > 0$ complex number lies in 1st quadrant and the value of θ is $0^0 \leq \theta \leq 90^0$.
$\theta = \pi/6$
$Z = 2 (\cos (\pi/6) + i \sin (\pi/6))$
∴ Polar form of $(\sqrt{3} + i)$ is $2 (\cos (\pi/6) + i \sin (\pi/6))$

(iii) 1 - i
Given: $Z = 1 - i$
So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{(1^2 + (-1)^2)}$
$= \sqrt{(1 + 1)}$
$= \sqrt{2}$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (1 / 1)$
$= \tan^{-1} 1$
Since $x > 0$, $y < 0$ complex number lies in 2nd quadrant and the value of θ is $-90^0 \leq \theta \leq 0^0$.
$\theta = -\pi/4$
$Z = \sqrt{2} (\cos (-\pi/4) + i \sin (-\pi/4))$
$= \sqrt{2} (\cos (\pi/4) - i \sin (\pi/4))$
∴ Polar form of $(1 - i)$ is $\sqrt{2} (\cos (\pi/4) - i \sin (\pi/4))$

(iv) (1 - i) / (1 + i)
Given: $Z = (1 - i) / (1 + i)$
Let us multiply and divide by (1 - i), we get

$$z = \frac{1-i}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{(1-i)^2}{1^2-i^2}$$
$$= \frac{1^2+i^2-2(1)(i)}{1-(-1)}$$
$$= \frac{1+(-1)-2i}{2}$$
$$= \frac{-2i}{2}$$
$$= 0 - i$$

So now,

$|Z| = \sqrt{(x^2 + y^2)}$

$= \sqrt{(0^2 + (-1)^2)}$

$= \sqrt{(0 + 1)}$

$= \sqrt{1}$

$\theta = \tan^{-1} (|y| / |x|)$

$= \tan^{-1} (1 / 0)$

$= \tan^{-1} \infty$

Since $x \geq 0$, $y < 0$ complex number lies in 4th quadrant and the value of θ is $-90^0 \leq \theta \leq 0^0$.

$\theta = -\pi/2$

$Z = 1 (\cos (-\pi/2) + i \sin (-\pi/2))$

$= 1 (\cos (\pi/2) - i \sin (\pi/2))$

∴ Polar form of $(1 - i) / (1 + i)$ is $1 (\cos (\pi/2) - i \sin (\pi/2))$

(v) $1/(1 + i)$

Given: $Z = 1 / (1 + i)$

Let us multiply and divide by $(1 - i)$, we get

$$Z = \frac{1}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{1-i}{1^2 - i^2}$$

$$= \frac{1-i}{1-(-1)}$$

$$= \frac{1-i}{2}$$

So now,

$|Z| = \sqrt{(x^2 + y^2)}$

$= \sqrt{((1/2)^2 + (-1/2)^2)}$

$= \sqrt{(1/4 + 1/4)}$

$= \sqrt{(2/4)}$

$= 1/\sqrt{2}$

$\theta = \tan^{-1} (|y| / |x|)$

$= \tan^{-1} ((1/2) / (1/2))$

$= \tan^{-1} 1$

Since $x > 0$, $y < 0$ complex number lies in 4th quadrant and the value of θ is $-90^0 \leq \theta \leq 0^0$.

$\theta = -\pi/4$

$Z = 1/\sqrt{2} (\cos (-\pi/4) + i \sin (-\pi/4))$

$= 1/\sqrt{2} (\cos (\pi/4) - i \sin (\pi/4))$

∴ Polar form of $1/(1 + i)$ is $1/\sqrt{2} (\cos (\pi/4) - i \sin (\pi/4))$

(vi) (1 + 2i) / (1 – 3i)
Given: Z = (1 + 2i) / (1 – 3i)
Let us multiply and divide by (1 + 3i), we get

$$Z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$
$$= \frac{1(1+3i)+2i(1+3i)}{1^2-(3i)^2}$$
$$= \frac{1+3i+2i+6i^2}{1-9i^2}$$
$$= \frac{1+5i+6(-1)}{1-9(-1)}$$
$$= \frac{-5+5i}{10}$$
$$= \frac{-1+i}{2}$$

So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{((-1/2)^2 + (1/2)^2)}$
$= \sqrt{(1/4 + 1/4)}$
$= \sqrt{(2/4)}$
$= 1/\sqrt{2}$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} ((1/2) / (1/2))$
$= \tan^{-1} 1$
Since x < 0, y > 0 complex number lies in 2nd quadrant and the value of θ is $90^0 \le \theta \le 180^0$.
$\theta = 3\pi/4$
$Z = 1/\sqrt{2} (\cos (3\pi/4) + i \sin (3\pi/4))$
∴ Polar form of (1 + 2i) / (1 – 3i) is $1/\sqrt{2} (\cos (3\pi/4) + i \sin (3\pi/4))$

(vii) sin 120° – i cos 120°
Given: Z = sin 120° – i cos 120°
$= \sqrt{3}/2 – i (-1/2)$
$= \sqrt{3}/2 + i (1/2)$
So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{((\sqrt{3}/2)^2 + (1/2)^2)}$
$= \sqrt{(3/4 + 1/4)}$
$= \sqrt{(4/4)}$
$= \sqrt{1}$
$= 1$

$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} ((1/2) / (\sqrt{3}/2))$
$= \tan^{-1} (1/\sqrt{3})$
Since x > 0, y > 0 complex number lies in 1st quadrant and the value of θ is $0^0 \leq \theta \leq 90^0$.
$\theta = \pi/6$
$Z = 1 (\cos (\pi/6) + i \sin (\pi/6))$
∴ Polar form of $\sqrt{3}/2 + i (1/2)$ is $1 (\cos (\pi/6) + i \sin (\pi/6))$

(viii) $-16 / (1 + i\sqrt{3})$
Given: $Z = -16 / (1 + i\sqrt{3})$
Let us multiply and divide by $(1 - i\sqrt{3})$, we get

$$Z = \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$
$$= \frac{-16+i16\sqrt{3}}{1^2-(i\sqrt{3})^2}$$
$$= \frac{-16+i16\sqrt{3}}{1-3i^2}$$
$$= \frac{-16+i16\sqrt{3}}{1-3(-1)}$$
$$= \frac{-16+i16\sqrt{3}}{4}$$
$$= -4 + i\,4\sqrt{3}$$

So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{((-4)^2 + (4\sqrt{3})^2)}$
$= \sqrt{(16 + 48)}$
$= \sqrt{(64)}$
$= 8$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} ((4\sqrt{3}) / 4)$
$= \tan^{-1} (\sqrt{3})$
Since x < 0, y > 0 complex number lies in 2nd quadrant and the value of θ is $90^0 \leq \theta \leq 180^0$.
$\theta = 2\pi/3$
$Z = 8 (\cos (2\pi/3) + i \sin (2\pi/3))$
∴ Polar form of $-16 / (1 + i\sqrt{3})$ is $8 (\cos (2\pi/3) + i \sin (2\pi/3))$

2. Write $(i^{25})^3$ in polar form.
Solution:
Given: $Z = (i^{25})^3$
$= i^{75}$

$= i^{74}. i$
$= (i^2)^{37}. i$
$= (-1)^{37}. i$
$= (-1). i$
$= - i$
$= 0 – i$

So now,
$|Z| = \sqrt{(x^2 + y^2)}$
$= \sqrt{(0^2 + (-1)^2)}$
$= \sqrt{(0 + 1)}$
$= \sqrt{1}$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (1 / 0)$
$= \tan^{-1} \infty$
Since $x \geq 0$, $y < 0$ complex number lies in 4th quadrant and the value of θ is $-90^0 \leq \theta \leq 0^0$.
$\theta = -\pi/2$
$Z = 1 (\cos (-\pi/2) + i \sin (-\pi/2))$
$= 1 (\cos (\pi/2) - i \sin (\pi/2))$
∴ Polar form of $(i^{25})^3$ is $1 (\cos (\pi/2) - i \sin (\pi/2))$

3. Express the following complex numbers in the form r (cos θ + i sin θ):
(i) 1 + i tan α
(ii) tan α – i
(iii) 1 – sin α + i cos α
(iv) (1 - i) / (cos π/3 + i sin π/3)
Solution:
(i) 1 + i tan α
Given: $Z = 1 + i \tan \alpha$
We know that the polar form of a complex number $Z = x + iy$ is given by $Z = |Z| (\cos \theta + i \sin \theta)$
Where,
$|Z|$ = modulus of complex number = $\sqrt{(x^2 + y^2)}$
$\theta = \arg (z)$ = argument of complex number = $\tan^{-1} (|y| / |x|)$
We also know that tan α is a periodic function with period π.
So α is lying in the interval $[0, \pi/2) \cup (\pi/2, \pi]$.

<u>Let us consider case 1:</u>
$\alpha \in [0, \pi/2)$
So now,

$|Z| = r = \sqrt{(x^2 + y^2)}$
$= \sqrt{(1^2 + \tan^2 \alpha)}$
$= \sqrt{(\sec^2 \alpha)}$
$= |\sec \alpha|$ since, sec α is positive in the interval [0, π/2)
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (\tan \alpha / 1)$
$= \tan^{-1} (\tan \alpha)$
= α since, tan α is positive in the interval [0, π/2)
∴ Polar form is Z = sec α (cos α + i sin α)

Let us consider case 2:
α ∈ (π/2, π]
So now,
$|Z| = r = \sqrt{(x^2 + y^2)}$
$= \sqrt{(1^2 + \tan^2 \alpha)}$
$= \sqrt{(\sec^2 \alpha)}$
$= |\sec \alpha|$
= - sec α since, sec α is negative in the interval (π/2, π]
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (\tan \alpha / 1)$
$= \tan^{-1} (\tan \alpha)$
= -π + α since, tan α is negative in the interval (π/2, π]
θ = -π + α [since, θ lies in 4^{th} quadrant]
Z = -sec α (cos (α - π) + i sin (α - π))
∴ Polar form is Z = -sec α (cos (α - π) + i sin (α - π))

(ii) tan α – i
Given: Z = tan α – i
We know that the polar form of a complex number Z = x + iy is given by Z = |Z| (cos θ + i sin θ)
Where,
$|Z|$ = modulus of complex number = $\sqrt{(x^2 + y^2)}$
θ = arg (z) = argument of complex number = $\tan^{-1} (|y| / |x|)$
We also know that tan α is a periodic function with period π.
So α is lying in the interval [0, π/2) ∪ (π/2, π].

Let us consider case 1:
α ∈ [0, π/2)
So now,

$|Z| = r = \sqrt{(x^2 + y^2)}$
$= \sqrt{(\tan^2 \alpha + 1^2)}$
$= \sqrt{(\sec^2 \alpha)}$
$= |\sec \alpha|$ since, $\sec \alpha$ is positive in the interval $[0, \pi/2)$
$= \sec \alpha$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (1/\tan \alpha)$
$= \tan^{-1} (\cot \alpha)$ since, $\cot \alpha$ is positive in the interval $[0, \pi/2)$
$= \alpha - \pi/2$ [since, θ lies in 4th quadrant]
$Z = \sec \alpha (\cos (\alpha - \pi/2) + i \sin (\alpha - \pi/2))$
∴ Polar form is $Z = \sec \alpha (\cos (\alpha - \pi/2) + i \sin (\alpha - \pi/2))$

<u>Let us consider case 2:</u>
$\alpha \in (\pi/2, \pi]$
So now,
$|Z| = r = \sqrt{(x^2 + y^2)}$
$= \sqrt{(\tan^2 \alpha + 1^2)}$
$= \sqrt{(\sec^2 \alpha)}$
$= |\sec \alpha|$
$= - \sec \alpha$ since, $\sec \alpha$ is negative in the interval $(\pi/2, \pi]$
$\theta = \tan^{-1} (|y| / |x|)$
$= \tan^{-1} (1/\tan \alpha)$
$= \tan^{-1} (\cot \alpha)$
$= \pi/2 + \alpha$ since, $\cot \alpha$ is negative in the interval $(\pi/2, \pi]$
$\theta = \pi/2 + \alpha$ [since, θ lies in 3rd quadrant]
$Z = -\sec \alpha (\cos (\pi/2 + \alpha) + i \sin (\pi/2 + \alpha))$
∴ Polar form is $Z = -\sec \alpha (\cos (\pi/2 + \alpha) + i \sin (\pi/2 + \alpha))$

(iii) $1 - \sin \alpha + i \cos \alpha$
Given: $Z = 1 - \sin \alpha + i \cos \alpha$
By using the formulas,
$\sin^2 \theta + \cos^2 \theta = 1$
$\sin 2\theta = 2 \sin \theta \cos \theta$
$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
So,

$$z = \left(\sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\right) + i\left(\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)\right)$$

$$= \left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)^2 + i\left(\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)\right)$$

We know that the polar form of a complex number Z = x + iy is given by Z = |Z| (cos θ + i sin θ)
Where,
|Z| = modulus of complex number = $\sqrt{(x^2 + y^2)}$
θ = arg (z) = argument of complex number = $\tan^{-1}$ (|y| / |x|)
Now,

$$|z| = \sqrt{(1-\sin\alpha)^2 + \cos^2\alpha}$$
$$= \sqrt{1 + \sin^2\alpha - 2\sin\alpha + \cos^2\alpha}$$
$$= \sqrt{1 + 1 - 2\sin\alpha}$$
$$= \sqrt{(2)(1-\sin\alpha)}$$
$$= \sqrt{(2)\left(\sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\right)}$$
$$= \sqrt{(2)\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)^2}$$
$$= \left|\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\right|$$

$$\theta = \tan^{-1}\left(\frac{\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)}{\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)^2}\right)$$
$$= \tan^{-1}\left(\frac{\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\left(\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\right)}{\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)^2}\right)$$
$$= \tan^{-1}\left(\frac{\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)}\right)$$
$$= \tan^{-1}\left(\frac{\cos\left(\frac{\alpha}{2}\right)\left(1 + \tan\left(\frac{\alpha}{2}\right)\right)}{\cos\left(\frac{\alpha}{2}\right)\left(1 - \tan\left(\frac{\alpha}{2}\right)\right)}\right)$$
$$= \tan^{-1}\left(\frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\alpha}{2}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\alpha}{2}\right)}\right)$$
$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right)$$

We know that sine and cosine functions are periodic with period 2π
Here we have 3 intervals:
$0 \le \alpha \le \pi/2$
$\pi/2 \le \alpha \le 3\pi/2$
$3\pi/2 \le \alpha \le 2\pi$

Let us consider case 1:

In the interval $0 \leq \alpha \leq \pi/2$

Cos $(\alpha/2) > \sin(\alpha/2)$ and also $0 < \pi/4 + \alpha/2 < \pi/2$

So,

$$|z| = \left|\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\right|$$

$$= \sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)$$

$$\theta = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right)$$

$= \pi/4 + \alpha/2$ [since, θ lies in 1st quadrant]

∴ Polar form is $Z = \sqrt{2}(\cos(\alpha/2) - \sin(\alpha/2))(\cos(\pi/4 + \alpha/2) + i\sin(\pi/4 + \alpha/2))$

Let us consider case 2:

In the interval $\pi/2 \leq \alpha \leq 3\pi/2$

Cos $(\alpha/2) < \sin(\alpha/2)$ and also $\pi/2 < \pi/4 + \alpha/2 < \pi$

So,

$$|z| = \left|\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\right|$$

$$= -\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)$$

$$= \sqrt{2}\left(\sin\left(\frac{\alpha}{2}\right) - \cos\left(\frac{\alpha}{2}\right)\right)$$

$$\theta = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right)$$

$= \pi - [\pi/4 + \alpha/2]$ [since, θ lies in 4th quadrant]

$= 3\pi/4 - \alpha/2$

Since, $(1 - \sin\alpha) > 0$ and $\cos\alpha < 0$ [Z lies in 4th quadrant]

$= \alpha/2 - 3\pi/4$

∴ Polar form is $Z = -\sqrt{2}(\cos(\alpha/2) - \sin(\alpha/2))(\cos(\alpha/2 - 3\pi/4) + i\sin(\alpha/2 - 3\pi/4))$

Let us consider case 3:

In the interval $3\pi/2 \leq \alpha \leq 2\pi$

Cos $(\alpha/2) < \sin(\alpha/2)$ and also $\pi < \pi/4 + \alpha/2 < 5\pi/4$

So,

$$|z| = \left|\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)\right|$$
$$= -\sqrt{2}\left(\cos\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\right)$$
$$= \sqrt{2}\left(\sin\left(\frac{\alpha}{2}\right) - \cos\left(\frac{\alpha}{2}\right)\right)$$

$\theta = \tan^{-1}(\tan(\pi/4 + \alpha/2))$

$= \pi - (\pi/4 + \alpha/2)$ [since, θ lies in 1st quadrant and tan's period is π]

$= \alpha/2 - 3\pi/4$

∴ Polar form is $Z = -\sqrt{2}(\cos(\alpha/2) - \sin(\alpha/2))(\cos(\alpha/2 - 3\pi/4) + i\sin(\alpha/2 - 3\pi/4))$

(iv) (1 - i) / (cos π/3 + i sin π/3)

Given: Z = (1 - i) / (cos π/3 + i sin π/3)

Let us multiply and divide by (1 - i√3), we get

$$Z = \frac{1-i}{\frac{1}{2}+\frac{i\sqrt{3}}{2}}$$
$$= 2 \times \frac{1-i}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$
$$= 2 \times \frac{1+i^2\sqrt{3}-i(1+\sqrt{3})}{1-i^2 3}$$
$$= 2 \times \frac{\left(1+(-\sqrt{3})-i(1+\sqrt{3})\right)}{1-(-3)}$$
$$= 2 \times \frac{(1-\sqrt{3})-i(1+\sqrt{3})}{4}$$
$$= \frac{(1-\sqrt{3})-i(1+\sqrt{3})}{2}$$

We know that the polar form of a complex number Z = x + iy is given by Z = |Z| (cos θ + i sin θ)

Where,

|Z| = modulus of complex number = $\sqrt{(x^2 + y^2)}$

θ = arg (z) = argument of complex number = $\tan^{-1}(|y| / |x|)$

Now,

$$|z| = \sqrt{\left(\frac{1-\sqrt{3}}{2}\right)^2 + \left(\frac{-1-\sqrt{3}}{2}\right)^2}$$
$$= \sqrt{\frac{1+3-2\sqrt{3}+1+2+2\sqrt{3}}{4}}$$
$$= \sqrt{\frac{8}{4}}$$
$$= \sqrt{2}$$

$\theta = \tan^{-1}\left(\left|\frac{\frac{1+\sqrt{3}}{2}}{\frac{1-\sqrt{3}}{2}}\right|\right)$

$= \tan^{-1}\left(\left|\frac{1+\sqrt{3}}{1-\sqrt{3}}\right|\right)$

$= \tan^{-1}\left(\left|\frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}\right|\right)$

$= \tan^{-1}\left(\left|\frac{1+3+2\sqrt{3}}{1-3}\right|\right)$

$= \tan^{-1}\left(\frac{4+2\sqrt{3}}{2}\right)$

Since x < 0, y < 0 complex number lies in 3rd quadrant and the value of θ is $180^0 \leq \theta \leq -90^0$.

$= \tan^{-1}(2 + \sqrt{3})$

$= -7\pi/12$

$Z = \sqrt{2}(\cos(-7\pi/12) + i \sin(-7\pi/12))$

$= \sqrt{2}(\cos(7\pi/12) - i \sin(7\pi/12))$

∴ Polar form of $(1 - i) / (\cos \pi/3 + i \sin \pi/3)$ is $\sqrt{2}(\cos(7\pi/12) - i \sin(7\pi/12))$

4. If z_1 and z_2 are two complex number such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$

Solution:

Given:

$|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$

Let us assume $\arg(z_1) = \theta$

$\arg(z_2) = \pi - \theta$

We know that in the polar form, $z = |z|(\cos\theta + i \sin\theta)$

$z_1 = |z_1|(\cos\theta + i \sin\theta)$..................(i)

$z_2 = |z_2|(\cos(\pi - \theta) + i \sin(\pi - \theta))$

$= |z_2|(-\cos\theta + i \sin\theta)$

$= -|z_2|(\cos\theta - i \sin\theta)$

Now let us find the conjugate of

$\bar{z}_2 = -|z_2|(\cos\theta + i \sin\theta)$ …… (ii) (since, $|\bar{z}_2| = |z_2|$)

Now,

$z_1 / \bar{z}_2 = [|z_1|(\cos\theta + i \sin\theta)] / [-|z_2|(\cos\theta + i \sin\theta)]$

$= -|z_1| / |z_2|$ [since, $|z_1| = |z_2|$]

$= -1$

When we cross multiply we get,

$z_1 = -\bar{z}_2$

Hence proved.

5. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, prove that arg (z_1/z_4) + arg (z_2/z_3) = 0

Solution:

Given:

$z_1 = \bar{z}_2$

$z_3 = \bar{z}_4$

We know that arg (z_1/z_2) = arg (z_1) – arg (z_2)

So,

arg (z_1/z_4) + arg (z_2/z_3) = arg (z_1) – arg (z_4) + arg (z_2) – arg (z_3)

= arg ($\bar{z}_2$) – arg (z_4) + arg (z_2) – arg ($\bar{z}_4$)

= [arg (z_2) + arg ($\bar{z}_2$)] – [arg (z_4) + arg ($\bar{z}_4$)]

= 0 – 0 [since, arg (z) + arg ($\bar{z}$) = 0]

= 0

Hence proved.

6. Express sin π/5 + i (1 – cos π/5) in polar form.

Solution:

Given:

Z = sin π/5 + i (1 – cos π/5)

By using the formula,

sin 2θ = 2 sin θ cos θ

1- cos 2θ = 2 $\sin^2$ θ

So,

Z = 2 sin π/10 cos π/10 + i (2 $\sin^2$ π/10)

= 2 sin π/10 (cos π/10 + i sin π/10)

∴ The polar form of sin π/5 + i (1 – cos π/5) is 2 sin π/10 (cos π/10 + i sin π/10)