

EXERCISE 11.1

Find the general solutions of the following equations:

(i) $\sin x = 1/2$

(ii) $\cos x = -\sqrt{3}/2$

(iii) $\operatorname{cosec} x = -\sqrt{2}$

(iv) $\sec x = \sqrt{2}$

(v) $\tan x = -1/\sqrt{3}$

(vi) $\sqrt{3} \sec x = 2$

Solution:

The general solution of any trigonometric equation is given as:

$\sin x = \sin y$, implies $x = n\pi + (-1)^n y$, where $n \in \mathbb{Z}$.

$\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbb{Z}$.

$\tan x = \tan y$, implies $x = n\pi + y$, where $n \in \mathbb{Z}$.

(i) $\sin x = 1/2$

We know $\sin 30^\circ = \sin \pi/6 = 1/2$

So,

$\sin x = \sin \pi/6$

\therefore the general solution is

$x = n\pi + (-1)^n \pi/6$, where $n \in \mathbb{Z}$. [since, $\sin x = \sin A \Rightarrow x = n\pi + (-1)^n A$]

(ii) $\cos x = -\sqrt{3}/2$

We know, $\cos 150^\circ = (-\sqrt{3}/2) = \cos 5\pi/6$

So,

$\cos x = \cos 5\pi/6$

\therefore the general solution is

$x = 2n\pi \pm 5\pi/6$, where $n \in \mathbb{Z}$.

(iii) $\operatorname{cosec} x = -\sqrt{2}$

Let us simplify,

$1/\sin x = -\sqrt{2}$ [since, $\operatorname{cosec} x = 1/\sin x$]

$\sin x = -1/\sqrt{2}$

$= \sin [\pi + \pi/4]$

$= \sin 5\pi/4$ or $\sin (-\pi/4)$

\therefore the general solution is

$x = n\pi + (-1)^{n+1} \pi/4$, where $n \in \mathbb{Z}$.

(iv) $\sec x = \sqrt{2}$

Let us simplify,

$$1/\cos x = \sqrt{2} \text{ [since, } \sec x = 1/\cos x \text{]}$$

$$\begin{aligned} \cos x &= 1/\sqrt{2} \\ &= \cos \pi/4 \end{aligned}$$

∴ the general solution is

$$x = 2n\pi \pm \pi/4, \text{ where } n \in \mathbb{Z}.$$

(v) $\tan x = -1/\sqrt{3}$

Let us simplify,

$$\tan x = -1/\sqrt{3}$$

$$\begin{aligned} \tan x &= \tan (\pi/6) \\ &= \tan (-\pi/6) \text{ [since, } \tan (-x) = -\tan x \text{]} \end{aligned}$$

∴ the general solution is

$$x = n\pi + (-\pi/6), \text{ where } n \in \mathbb{Z}.$$

$$\text{or } x = n\pi - \pi/6, \text{ where } n \in \mathbb{Z}.$$

(vi) $\sqrt{3} \sec x = 2$

Let us simplify,

$$\sec x = 2/\sqrt{3}$$

$$1/\cos x = 2/\sqrt{3}$$

$$\begin{aligned} \cos x &= \sqrt{3}/2 \\ &= \cos (\pi/6) \end{aligned}$$

∴ the general solution is

$$x = 2n\pi \pm \pi/6, \text{ where } n \in \mathbb{Z}.$$

2. Find the general solutions of the following equations:

(i) $\sin 2x = \sqrt{3}/2$

(ii) $\cos 3x = 1/2$

(iii) $\sin 9x = \sin x$

(iv) $\sin 2x = \cos 3x$

(v) $\tan x + \cot 2x = 0$

(vi) $\tan 3x = \cot x$

(vii) $\tan 2x \tan x = 1$

(viii) $\tan mx + \cot nx = 0$

(ix) $\tan px = \cot qx$

(x) $\sin 2x + \cos x = 0$

(xi) $\sin x = \tan x$

(xii) $\sin 3x + \cos 2x = 0$

Solution:

The general solution of any trigonometric equation is given as:

$$\sin x = \sin y, \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbb{Z}.$$

$$\cos x = \cos y, \text{ implies } x = 2n\pi \pm y, \text{ where } n \in \mathbb{Z}.$$

$$\tan x = \tan y, \text{ implies } x = n\pi + y, \text{ where } n \in \mathbb{Z}.$$

(i) $\sin 2x = \sqrt{3}/2$

Let us simplify,

$$\begin{aligned}\sin 2x &= \sqrt{3}/2 \\ &= \sin (\pi/3)\end{aligned}$$

\therefore the general solution is

$$2x = n\pi + (-1)^n \pi/3, \text{ where } n \in \mathbb{Z}.$$

$$x = n\pi/2 + (-1)^n \pi/6, \text{ where } n \in \mathbb{Z}.$$

(ii) $\cos 3x = 1/2$

Let us simplify,

$$\begin{aligned}\cos 3x &= 1/2 \\ &= \cos (\pi/3)\end{aligned}$$

\therefore the general solution is

$$3x = 2n\pi \pm \pi/3, \text{ where } n \in \mathbb{Z}.$$

$$x = 2n\pi/3 \pm \pi/9, \text{ where } n \in \mathbb{Z}.$$

(iii) $\sin 9x = \sin x$

Let us simplify,

$$\sin 9x - \sin x = 0$$

Using transformation formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

So,

$$\begin{aligned}&= 2 \cos (9x+x)/2 \sin (9x-x)/2 \\ &\Rightarrow \cos 5x \sin 4x = 0\end{aligned}$$

$$\cos 5x = 0 \text{ or } \sin 4x = 0$$

Let us verify both the expressions,

$$\cos 5x = 0$$

$$\cos 5x = \cos \pi/2$$

$$5x = (2n + 1)\pi/2$$

$$x = (2n + 1)\pi/10, \text{ where } n \in \mathbb{Z}.$$

$$\sin 4x = 0$$

$$\sin 4x = \sin 0$$

$$4x = n\pi$$

$$x = n\pi/4, \text{ where } n \in \mathbb{Z}.$$

\therefore the general solution is

$$x = (2n + 1)\pi/10 \text{ or } n\pi/4, \text{ where } n \in \mathbb{Z}.$$

(iv) $\sin 2x = \cos 3x$

Let us simplify,

$$\sin 2x = \cos 3x$$

$$\cos (\pi/2 - 2x) = \cos 3x \text{ [since, } \sin A = \cos (\pi/2 - A)\text{]}$$

$$\pi/2 - 2x = 2n\pi \pm 3x$$

$$\pi/2 - 2x = 2n\pi + 3x \text{ [or] } \pi/2 - 2x = 2n\pi - 3x$$

$$5x = \pi/2 + 2n\pi \text{ [or] } x = 2n\pi - \pi/2$$

$$5x = \pi/2 (1 + 4n) \text{ [or] } x = \pi/2 (4n - 1)$$

$$x = \pi/10 (1 + 4n) \text{ [or] } x = \pi/2 (4n - 1)$$

\therefore the general solution is

$$x = \pi/10 (4n + 1) \text{ [or] } x = \pi/2 (4n - 1), \text{ where } n \in \mathbb{Z}.$$

(v) $\tan x + \cot 2x = 0$

Let us simplify,

$$\tan x = -\cot 2x$$

$$\tan x = -\tan (\pi/2 - 2x) \text{ [since, } \cot A = \tan (\pi/2 - A)\text{]}$$

$$\tan x = \tan (2x - \pi/2) \text{ [since, } -\tan A = \tan -A\text{]}$$

$$x = n\pi + 2x - \pi/2$$

$$x = n\pi - \pi/2$$

\therefore the general solution is

$$x = n\pi - \pi/2, \text{ where } n \in \mathbb{Z}.$$

(vi) $\tan 3x = \cot x$

Let us simplify,

$$\tan 3x = \cot x$$

$$\tan 3x = \tan (\pi/2 - x) \text{ [since, } \cot A = \tan (\pi/2 - A)\text{]}$$

$$3x = n\pi + \pi/2 - x$$

$$4x = n\pi + \pi/2$$

$$x = n\pi/4 + \pi/8$$

\therefore the general solution is

$$x = n\pi/4 + \pi/8, \text{ where } n \in \mathbb{Z}.$$

(vii) $\tan 2x \tan x = 1$

Let us simplify,

$$\tan 2x \tan x = 1$$

$$\begin{aligned}\tan 2x &= 1/\tan x \\ &= \cot x\end{aligned}$$

$$\tan 2x = \tan (\pi/2 - x) \text{ [since, } \cot A = \tan (\pi/2 - A)\text{]}$$

$$2x = n\pi + \pi/2 - x$$

$$3x = n\pi + \pi/2$$

$$x = n\pi/3 + \pi/6$$

∴ the general solution is

$$x = n\pi/3 + \pi/6, \text{ where } n \in \mathbb{Z}.$$

(viii) $\tan mx + \cot nx = 0$

Let us simplify,

$$\tan mx + \cot nx = 0$$

$$\tan mx = -\cot nx$$

$$= -\tan (\pi/2 - nx) \text{ [since, } \cot A = \tan (\pi/2 - A)\text{]}$$

$$\tan mx = \tan (nx + \pi/2) \text{ [since, } -\tan A = \tan -A\text{]}$$

$$mx = k\pi + nx + \pi/2$$

$$(m - n)x = k\pi + \pi/2$$

$$(m - n)x = \pi (2k + 1)/2$$

$$x = \pi (2k + 1)/2(m - n)$$

∴ the general solution is

$$x = \pi (2k + 1)/2(m - n), \text{ where } m, n, k \in \mathbb{Z}.$$

(ix) $\tan px = \cot qx$

Let us simplify,

$$\tan px = \cot qx$$

$$\tan px = \tan (\pi/2 - qx) \text{ [since, } \cot A = \tan (\pi/2 - A)\text{]}$$

$$px = n\pi \pm (\pi/2 - qx)$$

$$(p + q)x = n\pi + \pi/2$$

$$x = n\pi/(p+q) + \pi/2(p+q)$$

$$= \pi (2n + 1)/2(p+q)$$

∴ the general solution is

$$x = \pi (2n + 1)/2(p+q), \text{ where } n \in \mathbb{Z}.$$

(x) $\sin 2x + \cos x = 0$

Let us simplify,

$$\sin 2x + \cos x = 0$$

$$\cos x = -\sin 2x$$

$$\cos x = -\cos (\pi/2 - 2x) \text{ [since, } \sin A = \cos (\pi/2 - A)\text{]}$$

$$= \cos (\pi - (\pi/2 - 2x)) \text{ [since, } -\cos A = \cos (\pi - A)\text{]}$$

$$= \cos(\pi/2 + 2x)$$

$$x = 2n\pi \pm (\pi/2 + 2x)$$

So,

$$x = 2n\pi + (\pi/2 + 2x) \text{ [or] } x = 2n\pi - (\pi/2 + 2x)$$

$$x = -\pi/2 - 2n\pi \text{ [or] } 3x = 2n\pi - \pi/2$$

$$x = -\pi/2(1 + 4n) \text{ [or] } x = \pi/6(4n - 1)$$

\therefore the general solution is

$$x = -\pi/2(1 + 4n), \text{ where } n \in \mathbb{Z}. \text{ [or] } x = \pi/6(4n - 1)$$

$$x = \pi/2(4n - 1), \text{ where } n \in \mathbb{Z}. \text{ [or] } x = \pi/6(4n - 1), \text{ where } n \in \mathbb{Z}.$$

(xi) $\sin x = \tan x$

Let us simplify,

$$\sin x = \tan x$$

$$\sin x = \sin x / \cos x$$

$$\sin x \cos x = \sin x$$

$$\sin x (\cos x - 1) = 0$$

So,

$$\sin x = 0 \text{ or } \cos x - 1 = 0$$

$$\sin x = \sin 0 \text{ [or] } \cos x = 1$$

$$\sin x = \sin 0 \text{ [or] } \cos x = \cos 0$$

$$x = n\pi \text{ [or] } x = 2m\pi$$

\therefore the general solution is

$$x = n\pi \text{ [or] } 2m\pi, \text{ where } n, m \in \mathbb{Z}.$$

(xii) $\sin 3x + \cos 2x = 0$

Let us simplify,

$$\sin 3x + \cos 2x = 0$$

$$\cos 2x = -\sin 3x$$

$$\cos 2x = -\cos(\pi/2 - 3x) \text{ [since, } \sin A = \cos(\pi/2 - A)\text{]}$$

$$\cos 2x = \cos(\pi - (\pi/2 - 3x)) \text{ [since, } -\cos A = \cos(\pi - A)\text{]}$$

$$\cos 2x = \cos(\pi/2 + 3x)$$

$$2x = 2n\pi \pm (\pi/2 + 3x)$$

So,

$$2x = 2n\pi + (\pi/2 + 3x) \text{ [or] } 2x = 2n\pi - (\pi/2 + 3x)$$

$$x = -\pi/2 - 2n\pi \text{ [or] } 5x = 2n\pi - \pi/2$$

$$x = -\pi/2(1 + 4n) \text{ [or] } x = \pi/10(4n - 1)$$

$$x = -\pi/2(4n + 1) \text{ [or] } \pi/10(4n - 1)$$

\therefore the general solution is

$$x = -\pi/2(4n + 1) \text{ [or] } \pi/10(4n - 1)$$

$$x = \pi/2 (4n - 1) \text{ [or] } \pi/10 (4n - 1), \text{ where } n \in \mathbb{Z}.$$

3. Solve the following equations:

(i) $\sin^2 x - \cos x = 1/4$

(ii) $2 \cos^2 x - 5 \cos x + 2 = 0$

(iii) $2 \sin^2 x + \sqrt{3} \cos x + 1 = 0$

(iv) $4 \sin^2 x - 8 \cos x + 1 = 0$

(v) $\tan^2 x + (1 - \sqrt{3}) \tan x - \sqrt{3} = 0$

(vi) $3 \cos^2 x - 2\sqrt{3} \sin x \cos x - 3 \sin^2 x = 0$

(vii) $\cos 4x = \cos 2x$

Solution:

The general solution of any trigonometric equation is given as:

$\sin x = \sin y$, implies $x = n\pi + (-1)^n y$, where $n \in \mathbb{Z}$.

$\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbb{Z}$.

$\tan x = \tan y$, implies $x = n\pi + y$, where $n \in \mathbb{Z}$.

(i) $\sin^2 x - \cos x = 1/4$

Let us simplify,

$$\sin^2 x - \cos x = 1/4$$

$$1 - \cos^2 x - \cos x = 1/4 \text{ [since, } \sin^2 x = 1 - \cos^2 x \text{]}$$

$$4 - 4 \cos^2 x - 4 \cos x = 1$$

$$4 \cos^2 x + 4 \cos x - 3 = 0$$

Let $\cos x$ be 'k'

So,

$$4k^2 + 4k - 3 = 0$$

$$4k^2 - 2k + 6k - 3 = 0$$

$$2k(2k - 1) + 3(2k - 1) = 0$$

$$(2k - 1) + (2k + 3) = 0$$

$$(2k - 1) = 0 \text{ or } (2k + 3) = 0$$

$$k = 1/2 \text{ or } k = -3/2$$

$$\cos x = 1/2 \text{ or } \cos x = -3/2$$

we shall consider only $\cos x = 1/2$. $\cos x = -3/2$ is not possible.

so,

$$\cos x = \cos 60^\circ = \cos \pi/3$$

$$x = 2n\pi \pm \pi/3$$

\therefore the general solution is

$$x = 2n\pi \pm \pi/3, \text{ where } n \in \mathbb{Z}.$$

(ii) $2 \cos^2 x - 5 \cos x + 2 = 0$

Let us simplify,

$$2 \cos^2 x - 5 \cos x + 2 = 0$$

Let $\cos x$ be 'k'

$$2k^2 - 5k + 2 = 0$$

$$2k^2 - 4k - k + 2 = 0$$

$$2k(k - 2) - 1(k - 2) = 0$$

$$(k - 2)(2k - 1) = 0$$

$$k = 2 \text{ or } k = 1/2$$

$$\cos x = 2 \text{ or } \cos x = 1/2$$

we shall consider only $\cos x = 1/2$. $\cos x = 2$ is not possible.

so,

$$\cos x = \cos 60^\circ = \cos \pi/3$$

$$x = 2n\pi \pm \pi/3$$

\therefore the general solution is

$$x = 2n\pi \pm \pi/3, \text{ where } n \in \mathbb{Z}.$$

(iii) $2 \sin^2 x + \sqrt{3} \cos x + 1 = 0$

Let us simplify,

$$2 \sin^2 x + \sqrt{3} \cos x + 1 = 0$$

$$2(1 - \cos^2 x) + \sqrt{3} \cos x + 1 = 0 \text{ [since, } \sin^2 x = 1 - \cos^2 x]$$

$$2 - 2 \cos^2 x + \sqrt{3} \cos x + 1 = 0$$

$$2 \cos^2 x - \sqrt{3} \cos x - 3 = 0$$

Let $\cos x$ be 'k'

$$2k^2 - \sqrt{3} k - 3 = 0$$

$$2k^2 - 2\sqrt{3} k + \sqrt{3} k - 3 = 0$$

$$2k(k - \sqrt{3}) + \sqrt{3}(k - \sqrt{3}) = 0$$

$$(2k + \sqrt{3})(k - \sqrt{3}) = 0$$

$$k = \sqrt{3} \text{ or } k = -\sqrt{3}/2$$

$$\cos x = \sqrt{3} \text{ or } \cos x = -\sqrt{3}/2$$

we shall consider only $\cos x = -\sqrt{3}/2$. $\cos x = \sqrt{3}$ is not possible.

so,

$$\cos x = -\sqrt{3}/2$$

$$\cos x = \cos 150^\circ = \cos 5\pi/6$$

$$x = 2n\pi \pm 5\pi/6, \text{ where } n \in \mathbb{Z}.$$

(iv) $4 \sin^2 x - 8 \cos x + 1 = 0$

Let us simplify,

$$4 \sin^2 x - 8 \cos x + 1 = 0$$

$$4(1 - \cos^2 x) - 8 \cos x + 1 = 0 \text{ [since, } \sin^2 x = 1 - \cos^2 x]$$

$$4 - 4 \cos^2 x - 8 \cos x + 1 = 0$$

$$4 \cos^2 x + 8 \cos x - 5 = 0$$

Let $\cos x$ be 'k'

$$4k^2 + 8k - 5 = 0$$

$$4k^2 - 2k + 10k - 5 = 0$$

$$2k(2k - 1) + 5(2k - 1) = 0$$

$$(2k + 5)(2k - 1) = 0$$

$$k = -5/2 = -2.5 \text{ or } k = 1/2$$

$$\cos x = -2.5 \text{ or } \cos x = 1/2$$

we shall consider only $\cos x = 1/2$. $\cos x = -2.5$ is not possible.

so,

$$\cos x = \cos 60^\circ = \cos \pi/3$$

$$x = 2n\pi \pm \pi/3$$

\therefore the general solution is

$$x = 2n\pi \pm \pi/3, \text{ where } n \in \mathbb{Z}.$$

$$(v) \tan^2 x + (1 - \sqrt{3}) \tan x - \sqrt{3} = 0$$

Let us simplify,

$$\tan^2 x + (1 - \sqrt{3}) \tan x - \sqrt{3} = 0$$

$$\tan^2 x + \tan x - \sqrt{3} \tan x - \sqrt{3} = 0$$

$$\tan x (\tan x + 1) - \sqrt{3} (\tan x + 1) = 0$$

$$(\tan x + 1) (\tan x - \sqrt{3}) = 0$$

$$\tan x = -1 \text{ or } \tan x = \sqrt{3}$$

As, $\tan x \in (-\infty, \infty)$ so both values are valid and acceptable.

$$\tan x = \tan (-\pi/4) \text{ or } \tan x = \tan (\pi/3)$$

$$x = m\pi - \pi/4 \text{ or } x = n\pi + \pi/3$$

\therefore the general solution is

$$x = m\pi - \pi/4 \text{ or } n\pi + \pi/3, \text{ where } m, n \in \mathbb{Z}.$$

$$(vi) 3 \cos^2 x - 2\sqrt{3} \sin x \cos x - 3 \sin^2 x = 0$$

Let us simplify,

$$3 \cos^2 x - 2\sqrt{3} \sin x \cos x - 3 \sin^2 x = 0$$

$$3 \cos^2 x - 3\sqrt{3} \sin x \cos x + \sqrt{3} \sin x \cos x - 3 \sin^2 x = 0$$

$$3 \cos x (\cos x - \sqrt{3} \sin x) + \sqrt{3} \sin x (\cos x - \sqrt{3} \sin x) = 0$$

$$\sqrt{3} (\cos x - \sqrt{3} \sin x) (\sqrt{3} \cos x + \sin x) = 0$$

$$\cos x - \sqrt{3} \sin x = 0 \text{ or } \sin x + \sqrt{3} \cos x = 0$$

$$\cos x = \sqrt{3} \sin x \text{ or } \sin x = -\sqrt{3} \cos x$$

$$\tan x = 1/\sqrt{3} \text{ or } \tan x = -\sqrt{3}$$

As, $\tan x \in (-\infty, \infty)$ so both values are valid and acceptable.

$\tan x = \tan (\pi/6)$ or $\tan x = \tan (-\pi/3)$
 $x = m\pi + \pi/6$ or $x = n\pi - \pi/3$
 \therefore the general solution is
 $x = m\pi + \pi/6$ or $n\pi - \pi/3$, where $m, n \in \mathbb{Z}$.

(vii) $\cos 4x = \cos 2x$

Let us simplify,

$$\cos 4x = \cos 2x$$

$$4x = 2n\pi \pm 2x$$

So,

$$4x = 2n\pi + 2x \text{ [or]} 4x = 2n\pi - 2x$$

$$2x = 2n\pi \text{ [or]} 6x = 2n\pi$$

$$x = n\pi \text{ [or]} x = n\pi/3$$

\therefore the general solution is

$$x = n\pi \text{ [or]} n\pi/3, \text{ where } n \in \mathbb{Z}.$$

4. Solve the following equations:

(i) $\cos x + \cos 2x + \cos 3x = 0$

(ii) $\cos x + \cos 3x - \cos 2x = 0$

(iii) $\sin x + \sin 5x = \sin 3x$

(iv) $\cos x \cos 2x \cos 3x = 1/4$

(v) $\cos x + \sin x = \cos 2x + \sin 2x$

(vi) $\sin x + \sin 2x + \sin 3x = 0$

(vii) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$

(viii) $\sin 3x - \sin x = 4 \cos^2 x - 2$

(ix) $\sin 2x - \sin 4x + \sin 6x = 0$

Solution:

The general solution of any trigonometric equation is given as:

$$\sin x = \sin y, \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbb{Z}.$$

$$\cos x = \cos y, \text{ implies } x = 2n\pi \pm y, \text{ where } n \in \mathbb{Z}.$$

$$\tan x = \tan y, \text{ implies } x = n\pi + y, \text{ where } n \in \mathbb{Z}.$$

(i) $\cos x + \cos 2x + \cos 3x = 0$

Let us simplify,

$$\cos x + \cos 2x + \cos 3x = 0$$

we shall rearrange and use transformation formula

$$\cos 2x + (\cos x + \cos 3x) = 0$$

$$\text{by using the formula, } \cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\cos 2x + 2 \cos (3x+x)/2 \cos (3x-x)/2 = 0$$

$$\begin{aligned} \cos 2x + 2\cos 2x \cos x &= 0 \\ \cos 2x (1 + 2\cos x) &= 0 \\ \cos 2x = 0 \text{ or } 1 + 2\cos x &= 0 \\ \cos 2x = \cos 0 \text{ or } \cos x &= -1/2 \\ \cos 2x = \cos \pi/2 \text{ or } \cos x &= \cos (\pi - \pi/3) \\ \cos 2x = \cos \pi/2 \text{ or } \cos x &= \cos (2\pi/3) \\ 2x = (2n + 1) \pi/2 \text{ or } x &= 2m\pi \pm 2\pi/3 \\ x = (2n + 1) \pi/4 \text{ or } x &= 2m\pi \pm 2\pi/3 \\ \therefore \text{ the general solution is} \\ x = (2n + 1) \pi/4 \text{ or } 2m\pi \pm 2\pi/3, &\text{ where } m, n \in \mathbb{Z}. \end{aligned}$$

(ii) $\cos x + \cos 3x - \cos 2x = 0$

Let us simplify,

$$\cos x + \cos 3x - \cos 2x = 0$$

we shall rearrange and use transformation formula

$$\cos x - \cos 2x + \cos 3x = 0$$

$$-\cos 2x + (\cos x + \cos 3x) = 0$$

By using the formula, $\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$

$$-\cos 2x + 2 \cos (3x+x)/2 \cos (3x-x)/2 = 0$$

$$-\cos 2x + 2\cos 2x \cos x = 0$$

$$\cos 2x (-1 + 2\cos x) = 0$$

$$\cos 2x = 0 \text{ or } -1 + 2\cos x = 0$$

$$\cos 2x = \cos 0 \text{ or } \cos x = 1/2$$

$$\cos 2x = \cos \pi/2 \text{ or } \cos x = \cos (\pi/3)$$

$$2x = (2n + 1) \pi/2 \text{ or } x = 2m\pi \pm \pi/3$$

$$x = (2n + 1) \pi/4 \text{ or } x = 2m\pi \pm \pi/3$$

\therefore the general solution is

$$x = (2n + 1) \pi/4 \text{ or } 2m\pi \pm \pi/3, \text{ where } m, n \in \mathbb{Z}.$$

(iii) $\sin x + \sin 5x = \sin 3x$

Let us simplify,

$$\sin x + \sin 5x = \sin 3x$$

$$\sin x + \sin 5x - \sin 3x = 0$$

we shall rearrange and use transformation formula

$$-\sin 3x + \sin x + \sin 5x = 0$$

$$-\sin 3x + (\sin 5x + \sin x) = 0$$

By using the formula, $\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$

$$-\sin 3x + 2 \sin (5x+x)/2 \cos (5x-x)/2 = 0$$

$$2\sin 3x \cos 2x - \sin 3x = 0$$

$$\begin{aligned}\sin 3x (2\cos 2x - 1) &= 0 \\ \sin 3x &= 0 \text{ or } 2\cos 2x - 1 = 0 \\ \sin 3x &= \sin 0 \text{ or } \cos 2x = 1/2 \\ \sin 3x &= \sin 0 \text{ or } \cos 2x = \cos \pi/3 \\ 3x &= n\pi \text{ or } 2x = 2m\pi \pm \pi/3 \\ x &= n\pi/3 \text{ or } x = m\pi \pm \pi/6 \\ \therefore \text{ the general solution is} \\ x &= n\pi/3 \text{ or } m\pi \pm \pi/6, \text{ where } m, n \in \mathbb{Z}.\end{aligned}$$

(iv) $\cos x \cos 2x \cos 3x = 1/4$

Let us simplify,

$$\cos x \cos 2x \cos 3x = 1/4$$

$$4 \cos x \cos 2x \cos 3x - 1 = 0$$

By using the formula,

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

$$2(2\cos x \cos 3x) \cos 2x - 1 = 0$$

$$2(\cos 4x + \cos 2x) \cos 2x - 1 = 0$$

$$2(2\cos^2 2x - 1 + \cos 2x) \cos 2x - 1 = 0 \text{ [using } \cos 2A = 2\cos^2 A - 1]$$

$$4\cos^3 2x - 2\cos 2x + 2\cos^2 2x - 1 = 0$$

$$2\cos^2 2x (2\cos 2x + 1) - 1(2\cos 2x + 1) = 0$$

$$(2\cos^2 2x - 1)(2\cos 2x + 1) = 0$$

So,

$$2\cos 2x + 1 = 0 \text{ or } (2\cos^2 2x - 1) = 0$$

$$\cos 2x = -1/2 \text{ or } \cos 4x = 0 \text{ [using } \cos 2\theta = 2\cos^2 \theta -$$

$$\cos 2x = \cos (\pi - \pi/3) \text{ or } \cos 4x = \cos \pi/2$$

$$\cos 2x = \cos 2\pi/3 \text{ or } \cos 4x = \cos \pi/2$$

$$2x = 2m\pi \pm 2\pi/3 \text{ or } 4x = (2n + 1) \pi/2$$

$$x = m\pi \pm \pi/3 \text{ or } x = (2n + 1) \pi/8$$

\therefore the general solution is

$$x = m\pi \pm \pi/3 \text{ or } (2n + 1) \pi/8, \text{ where } m, n \in \mathbb{Z}.$$

(v) $\cos x + \sin x = \cos 2x + \sin 2x$

Let us simplify,

$$\cos x + \sin x = \cos 2x + \sin 2x$$

upon rearranging we get,

$$\cos x - \cos 2x = \sin 2x - \sin x$$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

So,

$$-2 \sin (2x+x)/2 \sin (2x-x)/2 = 2 \cos (2x+x)/2 \sin (2x-x)/2$$

$$2 \sin 3x/2 \sin x/2 = 2 \cos 3x/2 \sin x/2$$

$$\sin x/2 (\sin 3x/2 - \cos 3x/2) = 0$$

So,

$$\sin x/2 = 0 \text{ or } \sin 3x/2 = \cos 3x/2$$

$$\sin x/2 = \sin m\pi \text{ or } \sin 3x/2 / \cos 3x/2 = 0$$

$$\sin x/2 = \sin m\pi \text{ or } \tan 3x/2 = 1$$

$$\sin x/2 = \sin m\pi \text{ or } \tan 3x/2 = \tan \pi/4$$

$$x/2 = m\pi \text{ or } 3x/2 = n\pi + \pi/4$$

$$x = 2m\pi \text{ or } x = 2n\pi/3 + \pi/6$$

∴ the general solution is

$$x = 2m\pi \text{ or } 2n\pi/3 + \pi/6, \text{ where } m, n \in \mathbb{Z}.$$

(vi) $\sin x + \sin 2x + \sin 3x = 0$

Let us simplify,

$$\sin x + \sin 2x + \sin 3x = 0$$

we shall rearrange and use transformation formula

$$\sin 2x + \sin x + \sin 3x = 0$$

By using the formula,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

So,

$$\sin 2x + 2 \sin (3x+x)/2 \cos (3x-x)/2 = 0$$

$$\sin 2x + 2 \sin 2x \cos x = 0$$

$$\sin 2x (2 \cos x + 1) = 0$$

$$\sin 2x = 0 \text{ or } 2 \cos x + 1 = 0$$

$$\sin 2x = \sin 0 \text{ or } \cos x = -1/2$$

$$\sin 2x = \sin 0 \text{ or } \cos x = \cos (\pi - \pi/3)$$

$$\sin 2x = \sin 0 \text{ or } \cos x = \cos 2\pi/3$$

$$2x = n\pi \text{ or } x = 2m\pi \pm 2\pi/3$$

$$x = n\pi/2 \text{ or } x = 2m\pi \pm 2\pi/3$$

∴ the general solution is

$$x = n\pi/2 \text{ or } 2m\pi \pm 2\pi/3, \text{ where } m, n \in \mathbb{Z}.$$

(vii) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$

Let us simplify,

$$\sin x + \sin 2x + \sin 3x + \sin 4x = 0$$

we shall rearrange and use transformation formula

$$\sin x + \sin 3x + \sin 2x + \sin 4x = 0$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So,

$$2 \sin \frac{(3x+x)}{2} \cos \frac{(3x-x)}{2} + 2 \sin \frac{(4x+2x)}{2} \cos \frac{(4x-2x)}{2} = 0$$

$$2 \sin 2x \cos x + 2 \sin 3x \cos x = 0$$

$$2 \cos x (\sin 2x + \sin 3x) = 0$$

Again by using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

we get,

$$2 \cos x (2 \sin \frac{(3x+2x)}{2} \cos \frac{(3x-2x)}{2}) = 0$$

$$2 \cos x (2 \sin \frac{5x}{2} \cos \frac{x}{2}) = 0$$

$$4 \cos x \sin \frac{5x}{2} \cos \frac{x}{2} = 0$$

So,

$$\cos x = 0 \text{ or } \sin \frac{5x}{2} = 0 \text{ or } \cos \frac{x}{2} = 0$$

$$\cos x = \cos 0 \text{ or } \sin \frac{5x}{2} = \sin 0 \text{ or } \cos \frac{x}{2} = \cos 0$$

$$\cos x = \cos \frac{\pi}{2} \text{ or } \sin \frac{5x}{2} = k\pi \text{ or } \cos \frac{x}{2} = \cos (2p + 1) \frac{\pi}{2}$$

$$x = (2n + 1) \frac{\pi}{2} \text{ or } \frac{5x}{2} = k\pi \text{ or } \frac{x}{2} = (2p + 1) \frac{\pi}{2}$$

$$x = (2n + 1) \frac{\pi}{2} \text{ or } x = \frac{2k\pi}{5} \text{ or } x = (2p + 1)\pi$$

$$x = n\pi + \frac{\pi}{2} \text{ or } x = \frac{2k\pi}{5} \text{ or } x = (2p + 1)\pi$$

\therefore the general solution is

$$x = n\pi + \frac{\pi}{2} \text{ or } x = \frac{2k\pi}{5} \text{ or } x = (2p + 1)\pi, \text{ where } n, k, p \in \mathbb{Z}.$$

(viii) $\sin 3x - \sin x = 4 \cos^2 x - 2$

Let us simplify,

$$\sin 3x - \sin x = 4 \cos^2 x - 2$$

$$\sin 3x - \sin x = 2(2 \cos^2 x - 1)$$

$$\sin 3x - \sin x = 2 \cos 2x \text{ [since, } \cos 2A = 2 \cos^2 A - 1 \text{]}$$

By using the formula,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

So,

$$2 \cos \frac{(3x+x)}{2} \sin \frac{(3x-x)}{2} = 2 \cos 2x$$

$$2 \cos 2x \sin x - 2 \cos 2x = 0$$

$$2 \cos 2x (\sin x - 1) = 0$$

Then,

$$2 \cos 2x = 0 \text{ or } \sin x - 1 = 0$$

$$\cos 2x = 0 \text{ or } \sin x = 1$$

$$\cos 2x = \cos 0 \text{ or } \sin x = \sin 1$$

$$\cos 2x = \cos 0 \text{ or } \sin x = \sin \frac{\pi}{2}$$

$$2x = (2n + 1) \frac{\pi}{2} \text{ or } x = m\pi + (-1)^m \frac{\pi}{2}$$

$$x = (2n + 1) \pi/4 \text{ or } x = m\pi + (-1)^m \pi/2$$

∴ the general solution is

$$x = (2n + 1) \pi/4 \text{ or } m\pi + (-1)^m \pi/2, \text{ where } m, n \in \mathbb{Z}.$$

(ix) $\sin 2x - \sin 4x + \sin 6x = 0$

Let us simplify,

$$\sin 2x - \sin 4x + \sin 6x = 0$$

we shall rearrange and use transformation formula

$$-\sin 4x + \sin 6x + \sin 2x = 0$$

By using the formula,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

we get,

$$-\sin 4x + 2 \sin (6x+2x)/2 \cos (6x-2x)/2 = 0$$

$$-\sin 4x + 2 \sin 4x \cos 2x = 0$$

$$\sin 4x (2 \cos 2x - 1) = 0$$

So,

$$\sin 4x = 0 \text{ or } 2 \cos 2x - 1 = 0$$

$$\sin 4x = \sin 0 \text{ or } \cos 2x = 1/2$$

$$\sin 4x = \sin 0 \text{ or } \cos 2x = \pi/3$$

$$4x = n\pi \text{ or } 2x = 2m\pi \pm \pi/3$$

$$x = n\pi/4 \text{ or } x = m\pi \pm \pi/6$$

∴ the general solution is

$$x = n\pi/4 \text{ or } m\pi \pm \pi/6, \text{ where } m, n \in \mathbb{Z}.$$

5. Solve the following equations:

(i) $\tan x + \tan 2x + \tan 3x = 0$

(ii) $\tan x + \tan 2x = \tan 3x$

(iii) $\tan 3x + \tan x = 2 \tan 2x$

Solution:

The general solution of any trigonometric equation is given as:

$$\sin x = \sin y, \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbb{Z}.$$

$$\cos x = \cos y, \text{ implies } x = 2n\pi \pm y, \text{ where } n \in \mathbb{Z}.$$

$$\tan x = \tan y, \text{ implies } x = n\pi + y, \text{ where } n \in \mathbb{Z}.$$

(i) $\tan x + \tan 2x + \tan 3x = 0$

Let us simplify,

$$\tan x + \tan 2x + \tan 3x = 0$$

$$\tan x + \tan 2x + \tan (x + 2x) = 0$$

By using the formula,

$$\tan (A+B) = [\tan A + \tan B] / [1 - \tan A \tan B]$$

So,

$$\tan x + \tan 2x + \frac{[\tan x + \tan 2x]}{[1 - \tan x \tan 2x]} = 0$$

$$(\tan x + \tan 2x) (1 + 1/(1 - \tan x \tan 2x)) = 0$$

$$(\tan x + \tan 2x) ([2 - \tan x \tan 2x] / [1 - \tan x \tan 2x]) = 0$$

Then,

$$(\tan x + \tan 2x) = 0 \text{ or } ([2 - \tan x \tan 2x] / [1 - \tan x \tan 2x]) = 0$$

$$(\tan x + \tan 2x) = 0 \text{ or } [2 - \tan x \tan 2x] = 0$$

$$\tan x = \tan (-2x) \text{ or } \tan x \tan 2x = 2$$

$$x = n\pi + (-2x) \text{ or } \tan x [2 \tan x / (1 - \tan^2 x)] = 2 \text{ [Using, } \tan 2x = 2 \tan x / 1 - \tan^2 x]$$

$$3x = n\pi \text{ or } 2 \tan^2 x / (1 - \tan^2 x) = 2$$

$$3x = n\pi \text{ or } 2 \tan^2 x = 2(1 - \tan^2 x)$$

$$3x = n\pi \text{ or } 2 \tan^2 x = 2 - 2 \tan^2 x$$

$$3x = n\pi \text{ or } 4 \tan^2 x = 2$$

$$x = n\pi/3 \text{ or } \tan^2 x = 2/4$$

$$x = n\pi/3 \text{ or } \tan^2 x = 1/2$$

$$x = n\pi/3 \text{ or } \tan x = 1/\sqrt{2}$$

$$x = n\pi/3 \text{ or } x = \tan^{-1} \alpha \text{ [let } 1/\sqrt{2} \text{ be '}\alpha\text{'}]$$

$$x = n\pi/3 \text{ or } x = m\pi + \alpha$$

\therefore the general solution is

$$x = n\pi/3 \text{ or } m\pi + \alpha, \text{ where } \alpha = \tan^{-1} 1/\sqrt{2}, m, n \in \mathbb{Z}.$$

(ii) $\tan x + \tan 2x = \tan 3x$

Let us simplify,

$$\tan x + \tan 2x = \tan 3x$$

$$\tan x + \tan 2x - \tan 3x = 0$$

$$\tan x + \tan 2x - \tan (x + 2x) = 0$$

By using the formula,

$$\tan (A+B) = [\tan A + \tan B] / [1 - \tan A \tan B]$$

So,

$$\tan x + \tan 2x - \frac{[\tan x + \tan 2x]}{[1 - \tan x \tan 2x]} = 0$$

$$(\tan x + \tan 2x) (1 - 1/(1 - \tan x \tan 2x)) = 0$$

$$(\tan x + \tan 2x) ([- \tan x \tan 2x] / [1 - \tan x \tan 2x]) = 0$$

Then,

$$(\tan x + \tan 2x) = 0 \text{ or } ([- \tan x \tan 2x] / [1 - \tan x \tan 2x]) = 0$$

$$(\tan x + \tan 2x) = 0 \text{ or } [- \tan x \tan 2x] = 0$$

$$\tan x = \tan (-2x) \text{ or } -\tan x \tan 2x = 0$$

$$\tan x = \tan (-2x) \text{ or } 2 \tan^2 x / (1 - \tan^2 x) = 0 \text{ [Using, } \tan 2x = 2 \tan x / 1 - \tan^2 x]$$

$$x = n\pi + (-2x) \text{ or } x = m\pi + 0$$

$$3x = n\pi \text{ or } x = m\pi$$

$$x = n\pi/3 \text{ or } x = m\pi$$

∴ the general solution is

$$x = n\pi/3 \text{ or } m\pi, \text{ where } m, n \in \mathbb{Z}.$$

(iii) $\tan 3x + \tan x = 2 \tan 2x$

Let us simplify,

$$\tan 3x + \tan x = 2 \tan 2x$$

$$\tan 3x + \tan x = \tan 2x + \tan 2x$$

upon rearranging we get,

$$\tan 3x - \tan 2x = \tan 2x - \tan x$$

By using the formula,

$$\tan (A-B) = [\tan A - \tan B] / [1 + \tan A \tan B]$$

so,

$$[(\tan 3x - \tan 2x) (1 + \tan 3x \tan 2x)] / [1 + \tan 3x \tan 2x] = [(\tan 2x - \tan x) (1 + \tan x \tan 2x)] / [1 + \tan 2x \tan x]$$

$$\tan (3x - 2x) (1 + \tan 3x \tan 2x) = \tan (2x - x) (1 + \tan x \tan 2x)$$

$$\tan x [1 + \tan 3x \tan 2x - 1 - \tan 2x \tan x] = 0$$

$$\tan x \tan 2x (\tan 3x - \tan x) = 0$$

so,

$$\tan x = 0 \text{ or } \tan 2x = 0 \text{ or } (\tan 3x - \tan x) = 0$$

$$\tan x = 0 \text{ or } \tan 2x = 0 \text{ or } \tan 3x = \tan x$$

$$x = n\pi \text{ or } 2x = m\pi \text{ or } 3x = k\pi + x$$

$$x = n\pi \text{ or } x = m\pi/2 \text{ or } 2x = k\pi$$

$$x = n\pi \text{ or } x = m\pi/2 \text{ or } x = k\pi/2$$

∴ the general solution is

$$x = n\pi \text{ or } m\pi/2 \text{ or } k\pi/2, \text{ where } m, n, k \in \mathbb{Z}.$$