

EXERCISE 12.1

1. Find the second order derivatives of the each of the following functions:

(i) $x^3 + \tan x$

Solution:

Given, $y = x^3 + \tan x$

We have to find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find $\frac{dy}{dx}$ and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (x^3 + \tan x) = \frac{d}{dx} (x^3) + \frac{d}{dx} (\tan x)$$

$$= 3x^2 + \sec^2 x$$

$$\therefore \frac{dy}{dx} = 3x^2 + \sec^2 x$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (3x^2 + \sec^2 x) = \frac{d}{dx} (3x^2) + \frac{d}{dx} (\sec^2 x)$$

$$\frac{d^2y}{dx^2} = 6x + 2 \sec x \sec x \tan x$$

$$\frac{d^2y}{dx^2} = 6x + 2 \sec^2 x \tan x$$

(ii) $\sin (\log x)$

Solution:

Let, $y = \sin (\log x)$

We have to find $\frac{d^2y}{dx^2}$

We know that $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\sin(\log x))$$

Differentiating $\sin(\log x)$ using the chain rule,

Let, $t = \log x$ and $y = \sin t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ [using chain rule]}$$

$$\frac{dy}{dx} = \cos t \times \frac{1}{x}$$

$$\frac{dy}{dx} = \cos(\log x) \times \frac{1}{x}$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\cos(\log x) \times \frac{1}{x} \right)$$

$$\frac{d^2y}{dx^2} = \cos(\log x) \times \frac{-1}{x^2} + \frac{1}{x} \times \frac{1}{x} (-\sin(\log x))$$

Now by using product rule for differentiation we get,

$$= \frac{-1}{x^2} \cos(\log x) - \frac{1}{x^2} \sin(\log x)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2} \cos(\log x) - \frac{1}{x^2} \sin(\log x)$$

(iii) $\log(\sin x)$

Solution:

Let, $y = \log(\sin x)$

We have to find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\log (\sin x))$$

Differentiating $\sin (\log x)$ using chain rule,

Let, $t = \sin x$ and $y = \log t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ [using chain rule]}$$

$$\frac{dy}{dx} = \cos x \times \frac{1}{t}$$

$$\left[\because \frac{d}{dx} \log x = \frac{1}{x} \text{ \& } \frac{d}{dx} (\sin x) = \cos x \right]$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\cot x)$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x \left[\because \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x \right]$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

(iv) $e^x \sin 5x$

Solution:

Let, $y = e^x \sin 5x$

Now we have to find $\frac{d^2y}{dx^2}$

We know, $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^x \sin 5x)$$

Let $u = e^x$ and $v = \sin 5x$

As, $y = uv$

Now by using product rule of differentiation:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = e^x \frac{d}{dx} (\sin 5x) + \sin 5x \frac{d}{dx} e^x$$

$$\frac{dy}{dx} = 5e^x \cos 5x + e^x \sin 5x$$

$$\left[\because \frac{d}{dx} (\sin ax) = a \cos ax, \text{ where } a \text{ is any constant \& } \frac{d}{dx} e^x = e^x \right]$$

Again differentiating with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (5e^x \cos 5x + e^x \sin 5x)$$

$$= \frac{d}{dx} (5e^x \cos 5x) + \frac{d}{dx} (e^x \sin 5x)$$

Again using the product rule

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx} (\sin 5x) + \sin 5x \frac{d}{dx} e^x + 5e^x \frac{d}{dx} (\cos 5x) + \cos 5x \frac{d}{dx} (5e^x)$$

$$\frac{d^2y}{dx^2} = 5e^x \cos 5x - 25e^x \sin 5x + e^x \sin 5x + 5e^x \cos 5x$$

$$\frac{d^2y}{dx^2} = 10e^x \cos 5x - 24e^x \sin 5x$$

(v) $e^{6x} \cos 3x$

Solution:

Let, $y = e^{6x} \cos 3x$

We have to find $\frac{d^2y}{dx^2}$

We know, $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^{6x} \cos 3x)$$

Let $u = e^{6x}$ and $v = \cos 3x$

We have, $y = uv$

Now by using product rule of differentiation

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = e^{6x} \frac{d}{dx} (\cos 3x) + \cos 3x \frac{d}{dx} e^{6x}$$

$$\frac{dy}{dx} = -3e^{6x} \sin 3x + 6e^{6x} \cos 3x$$

$$\left[\because \frac{d}{dx} (\cos ax) = -a \sin ax, a \text{ is any constant} \& \frac{d}{dx} e^{ax} = ae^{ax} \right]$$

Again differentiating with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (-3e^{6x} \sin 3x + 6e^{6x} \cos 3x)$$

$$= \frac{d}{dx} (-3e^{6x} \sin 3x) + \frac{d}{dx} (6e^{6x} \cos 3x)$$

Again using the product rule

$$\frac{d^2y}{dx^2} = -3e^{6x} \frac{d}{dx} (\sin 3x) - 3 \sin 3x \frac{d}{dx} e^{6x} + 6e^{6x} \frac{d}{dx} (\cos 3x) + \cos 3x \frac{d}{dx} (6e^{6x})$$

$$\frac{d^2y}{dx^2} = -9e^{6x} \cos 3x - 18e^{6x} \sin 3x - 18e^{6x} \sin 3x + 36e^{6x} \cos 3x$$

$$\frac{d^2y}{dx^2} = 27e^{6x} \cos 3x - 36e^{6x} \sin 3x$$

(vi) $x^3 \log x$

Solution:

Let, $y = x^3 \log x$

We have to find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (x^3 \log x)$$

Let $u = x^3$ and $v = \log x$

We have, $y = uv$

Now by using product rule of differentiation

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = x^3 \frac{d}{dx} (\log x) + \log x \frac{d}{dx} x^3$$

$$\frac{dy}{dx} = 3x^2 \log x + \frac{x^3}{x}$$

$$\left[\because \frac{d}{dx} (\log x) = \frac{1}{x} \text{ and } \frac{d}{dx} (x^n) = nx^{n-1} \right]$$

Again differentiating with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (3x^2 \log x + x^2)$$

$$= \frac{d}{dx} (3x^2 \log x) + \frac{d}{dx} (x^2)$$

Again using the product rule

$$\frac{d^2y}{dx^2} = 3 \log x \frac{d}{dx} x^2 + 3x^2 \frac{d}{dx} \log x + \frac{d}{dx} x^2$$

We know $\frac{d}{dx} (\log x) = \frac{1}{x}$ and $\frac{d}{dx} (x^n) = nx^{n-1}$

$$\frac{d^2y}{dx^2} = 6x \log x + \frac{3x^2}{x} + 2x$$

$$\frac{d^2y}{dx^2} = 6x \log x + 5x$$

(vii) $\tan^{-1}x$

Solution:

Let, $y = \tan^{-1}x$

We have to find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\tan^{-1}x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{1+x^2} \right)$$

Differentiating $\frac{1}{1+x^2}$ using chain rule,

Let $t = 1+x^2$ and $z = 1/t$

$$\therefore \frac{dz}{dx} = \frac{dz}{dt} \times \frac{dt}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{t^2} \times 2x = -\frac{2x}{1+x^2} \left[\because \frac{d}{dx}(x^n) = nx^{n-1} \right]$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{2x}{(1+x^2)^2}$$

(viii) $x \cos x$ **Solution:**

Let, $y = x \cos x$

We have to find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x \cos x)$$

Let $u = x$ and $v = \cos x$

As, $y = uv$

Now by using product rule of differentiation:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}x$$

$$\frac{dy}{dx} = -x \sin x + \cos x$$

$$\left[\because \frac{d}{dx}(\cos x) = -\sin x \text{ and } \frac{d}{dx}(x^n) = nx^{n-1} \right]$$

Again differentiating with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx}(-x \sin x + \cos x)$$

$$- \frac{d}{dx} (-x \sin x) + \frac{d}{dx} \cos x$$

Again using the product rule

$$\frac{d^2 y}{dx^2} = -x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} (-x) + \frac{d}{dx} \cos x$$

$$[\because \frac{d}{dx} (\sin x) = \cos x \text{ and } \frac{d}{dx} (x^n) = nx^{n-1}]$$

$$\frac{d^2 y}{dx^2} = -x \cos x - \sin x - \sin x$$

$$\frac{d^2 y}{dx^2} = -x \cos x - 2 \sin x$$

(ix) Log (log x)

Solution:

Let, $y = \log (\log x)$

We have to find $\frac{d^2 y}{dx^2}$

We know, $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\log \log x)$$

Let $y = \log t$ and $t = \log x$

Using chain rule of differentiation:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{t} \times \frac{1}{x} = \frac{1}{x \log x}$$

Again differentiating with respect to x :

$$\text{As, } \frac{dy}{dx} = u \times v$$

$$\text{Where } u = \frac{1}{x} \text{ and } v = \frac{1}{\log x}$$

Now by using product rule of differentiation:

$$\frac{d^2y}{dx^2} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{x} \frac{d}{dx} \left(\frac{1}{\log x} \right) + \frac{1}{\log x} \frac{d}{dx} \left(\frac{1}{x} \right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2 (\log x)^2} - \frac{1}{x^2 \log x}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{x^2 (\log x)^2} - \frac{1}{x^2 \log x}$$

2. If $y = e^{-x} \cos x$, show that $\frac{d^2y}{dx^2} = 2e^{-x} \sin x$.

Solution:

$$\text{Let } y = e^{-x} \cos x$$

We have to find $\frac{d^2y}{dx^2}$

$$\text{We have, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^{-x} \cos x)$$

$$\text{Let } u = e^{-x} \text{ and } v = \cos x$$

$$\text{We have, } y = u v$$

Differentiate the above by using product rule,

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = e^{-x} \frac{d}{dx}(\cos x) + \cos x \frac{dy}{dx} e^{-x}$$

$$\frac{dy}{dx} = -e^{-x} \sin x - e^{-x} \cos x$$

$$\left[\because \frac{d}{dx}(\cos x) = -\sin x \text{ \& \ } \frac{d}{dx} e^{-x} = -e^{-x} \right]$$

Again differentiating with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (-e^{-x} \sin x - e^{-x} \cos x)$$

$$= \frac{d}{dx} (-e^{-x} \sin x) - \frac{d}{dx} (e^{-x} \cos x)$$

Again by using product rule we get

$$\frac{d^2 y}{dx^2} = -e^{-x} \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx} e^{-x} - e^{-x} \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(e^{-x})$$

$$\frac{d^2 y}{dx^2} = -e^{-x} \cos x + e^{-x} \sin x + e^{-x} \sin x + e^{-x} \cos x$$

$$\left[\because \frac{d}{dx}(\cos x) = -\sin x, \frac{d}{dx} e^{-x} = -e^{-x} \right]$$

$$\frac{d^2 y}{dx^2} = 2e^{-x} \sin x$$

Hence proved.

3. If $y = x + \tan x$, show that $\cos^2 x \frac{d^2 y}{dx^2} - 2y + 2x = 0$.

Solution:

Given $y = x + \tan x$

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (x + \tan x) = \frac{d}{dx} (x) + \frac{d}{dx} (\tan x) = 1 + \sec^2 x$$

$$\therefore \frac{dy}{dx} = 1 + \sec^2 x$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (1 + \sec^2 x) = \frac{d}{dx} (1) + \frac{d}{dx} (\sec^2 x)$$

By using chain rule, we get

$$\frac{d^2y}{dx^2} = 0 + 2 \sec x \sec x \tan x$$

$$\frac{d^2y}{dx^2} = 2 \sec^2 x \tan x$$

As we got an expression for the second order, as we need $\cos^2 x$ term with $\frac{d^2y}{dx^2}$

Multiply both sides of equation 1 with $\cos^2 x$

We have,

$$\cos^2 x \frac{d^2y}{dx^2} = 2 \cos^2 x \sec^2 x \tan x \quad [\because \cos x \times \sec x = 1]$$

$$\cos^2 x \frac{d^2y}{dx^2} = 2 \tan x$$

From the given equation $\tan x = y - x$

$$\therefore \cos^2 x \frac{d^2y}{dx^2} = 2(y - x)$$

$$\therefore \cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$$

4. If $y = x^3 \log x$, prove that $\frac{d^4 y}{dx^4} = \frac{6}{x}$.

Solution:

Given, $y = x^3 \log x$

Let's find $\frac{d^4 y}{dx^4}$

$$\text{As } \frac{d^4 y}{dx^4} = \frac{d}{dx} \left(\frac{d^3 y}{dx^3} \right) = \frac{d}{dx} \frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right) = \frac{d}{dx} \left(\frac{d}{dx} \left(\frac{d}{dx} \left(\frac{dy}{dx} \right) \right) \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (x^3 \log x)$$

Again differentiating by using product rule, we get

$$\frac{dy}{dx} = x^3 \frac{d}{dx} \log x + \log x \frac{d}{dx} x^3$$

$$\frac{dy}{dx} = \frac{x^3}{x} + 3x^2 \log x$$

$$\frac{dy}{dx} = x^2(1 + 3 \log x)$$

Again differentiating using product rule:

$$\frac{d^2 y}{dx^2} = x^2 \frac{d}{dx} (1 + 3 \log x) + (1 + 3 \log x) \frac{d}{dx} x^2$$

$$\frac{d^2 y}{dx^2} = x^2 \times \frac{3}{x} + (1 + 3 \log x) \times 2x$$

$$\frac{d^2 y}{dx^2} = x(5 + 6 \log x)$$

Again differentiating using product rule

$$\frac{d^3 y}{dx^3} = x \frac{d}{dx} (5 + 6 \log x) + (5 + 6 \log x) \frac{d}{dx} x$$

$$\frac{d^3y}{dx^3} = x \times \frac{6}{x} + (5 + 6 \log x)$$

$$\frac{d^3y}{dx^3} = 11 + 6 \log x$$

Again differentiating with respect to x

$$\frac{d^4y}{dx^4} = \frac{6}{x}$$

Hence proved.

5. If $y = \log(\sin x)$, prove that $\frac{d^3y}{dx^3} = 2 \cos x \operatorname{cosec}^3 x$.

Solution:

Given, $y = \log(\sin x)$

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{d}{dx} \left(\frac{dy}{dx} \right) \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\log(\sin x))$$

Differentiating $\log(\sin x)$ using the chain rule,

Let, $t = \sin x$ and $y = \log t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ [using chain rule]}$$

$$\frac{dy}{dx} = \cos x \times \frac{1}{t}$$

$$\left[\because \frac{d}{dx} \log x = \frac{1}{x} \text{ \& } \frac{d}{dx} (\sin x) = \cos x \right]$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\cot x)$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

$$\left[\because \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x \right]$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{d}{dx} (-\operatorname{cosec}^2 x)$$

$$\text{Using the chain rule and } \frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$$

$$\frac{d^3y}{dx^3} = -2\operatorname{cosec} x (-\operatorname{cosec} x \cot x)$$

$$= 2\operatorname{cosec}^2 x \cot x = 2 \operatorname{cosec}^2 x \frac{\cos x}{\sin x}$$

$$\therefore \frac{d^3y}{dx^3} = 2\operatorname{cosec}^2 x \cos x$$

Hence proved.

6. If $y = 2 \sin x + 3 \cos x$, show that $\frac{d^2y}{dx^2} + y = 0$.

Solution:

$$\text{Given } y = 2 \sin x + 3 \cos x$$

$$\text{Let's find } \frac{d^2y}{dx^2}$$

$$\text{We know } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (2 \sin x + 3 \cos x) = 2 \frac{d}{dx} (\sin x) + 3 \frac{d}{dx} (\cos x)$$

$$= 2 \cos x - 3 \sin x$$

$$\therefore \frac{dy}{dx} = 2 \cos x - 3 \sin x$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (2 \cos x - 3 \sin x) = \frac{2d}{dx} \cos x - 3 \frac{d}{dx} \sin x$$

$$\frac{d^2y}{dx^2} = -2 \sin x - 3 \cos x$$

We have, $y = 2 \sin x + 3 \cos x$

$$\therefore \frac{d^2y}{dx^2} = -(2 \sin x + 3 \cos x) = -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence proved.

7. If $y = \frac{\log x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{2 \log x - 3}{x^3}$.

Solution:

Given $y = \log x/x$

Let's find $\frac{d^2y}{dx^2}$

$$\text{We know } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx and differentiate it again.

As y is the product of two functions u and v

Let $u = \log x$ and $v = 1/x$

Now by using product rule

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx} \left(\frac{\log x}{x} \right) = \log x \frac{d}{dx} \frac{1}{x} + \frac{1}{x} \frac{d}{dx} \log x$$

$$\frac{dy}{dx} = -\frac{1}{x^2} \log x + \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2} (1 - \log x)$$

Again using the product rule to find $\frac{d^2y}{dx^2}$

$$\begin{aligned} \frac{d^2y}{dx^2} &= (1 - \log x) \frac{d}{dx} \frac{1}{x^2} + \frac{1}{x^2} \frac{d}{dx} (1 - \log x) \\ &= -2 \left(\frac{1 - \log x}{x^3} \right) - \frac{1}{x^3} \end{aligned}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2 \log x - 3}{x^3}$$

Hence proved.

8. If $x = a \sec \theta$, $y = b \tan \theta$, prove that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2 y^3}$.

Solution:

If $y = f(\theta)$ and $x = g(\theta)$ that is y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

We can write:

Given,

$$x = a \sec \theta \text{equation 1}$$

$$y = b \tan \theta \text{equation 2}$$

$$\text{We have to prove } \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$$

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a \sec \theta = a \sec \theta \tan \theta \text{equation 3}$$

$$\text{Similarly, } \frac{dy}{d\theta} = b \sec^2 \theta \text{ equation 4}$$

$$[\because \frac{d}{dx} \sec x = \sec x \tan x, \frac{d}{dx} \tan x = \sec^2 x]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b}{a} \operatorname{cosec} \theta$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{b}{a} \operatorname{cosec} \theta \right)$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a} \operatorname{cosec} \theta \cot \theta \frac{d\theta}{dx} \text{equation 5 [using chain rule]}$$

From equation 3:

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta$$

$$\therefore \frac{d\theta}{dx} = \frac{1}{a \sec \theta \tan \theta}$$

Putting the value in equation 5

$$\frac{d^2y}{dx^2} = -\frac{b}{a} \operatorname{cosec} \theta \cot \theta \frac{1}{a \sec \theta \tan \theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2 \tan^3 \theta}$$

From equation 1:

$$y = b \tan \theta$$

$$\therefore \frac{d^3y}{dx^3} = \frac{-b}{a^2 y^2} = -\frac{b^3}{a^2 y^2}$$

9. If $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$, prove that $\frac{d^2x}{d\theta^2} = a(\cos \theta - \theta \sin \theta)$, $\frac{d^2y}{d\theta^2} = a(\sin \theta + \theta \cos \theta)$ and $\frac{d^2y}{dx^2} = \frac{\sec^3 \theta}{2a}$

Solution:

If $y = f(\theta)$ and $x = g(\theta)$, that is y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\text{We can write: } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Given,

$$x = a(\cos \theta + \theta \sin \theta) \dots \text{equation 1}$$

$$y = a(\sin \theta - \theta \cos \theta) \dots \text{equation 2}$$

Let's find $\frac{d^2y}{dx^2}$

$$\text{We know } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{d}{d\theta} a(\cos \theta + \theta \sin \theta) \\ &= a(-\sin \theta + \theta \cos \theta + \sin \theta) \\ &= a \theta \cos \theta \dots \text{Equation 4}\end{aligned}$$

Again differentiating with respect to θ using product rule

$$\begin{aligned}\frac{d^2x}{d\theta^2} &= a(-\theta \sin \theta + \cos \theta) \\ \therefore \frac{d^2x}{d\theta^2} &= a(\cos \theta - \theta \sin \theta)\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{d}{d\theta} a(\sin \theta - \theta \cos \theta) = a \frac{d}{d\theta} \sin \theta - a \frac{d}{d\theta} (\theta \cos \theta) \\ &= a \cos \theta + a \theta \sin \theta - a \cos \theta \\ \therefore \frac{dy}{d\theta} &= a \theta \sin \theta \dots \text{equation 5}\end{aligned}$$

Again differentiating with respect to θ using product rule

$$\begin{aligned}\frac{d^2y}{d\theta^2} &= a(\theta \cos \theta + \sin \theta) \\ \therefore \frac{d^2y}{d\theta^2} &= a(\sin \theta + \theta \cos \theta)\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Using equation 4 and 5, we have

$$\frac{dy}{dx} = \frac{a\theta \sin \theta}{a\theta \cos \theta} = \tan \theta$$

$$\text{We have } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

Again differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \tan \theta$$

$$= \sec^2 \theta \frac{d\theta}{dx}$$

$$\because \frac{dx}{d\theta} = a\theta \cos \theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{a\theta \cos \theta}$$

Putting a value in the above equation we get

$$\frac{d^2y}{dx^2} = \sec^2 \theta \times \frac{1}{a\theta \cos \theta}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^3 \theta}{a\theta}$$

10. If $y = e^x \cos x$, prove that $\frac{d^2y}{dx^2} = 2e^x \cos \left(x + \frac{\pi}{2}\right)$.

Solution:

Given, $y = e^x \cos x$

We have to find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^x \cos x)$$

Let $u = e^x$ and $v = \cos x$

As, $y = u v$

Now by using product rule we get

$$\therefore \frac{dy}{dx} = e^x \frac{d}{dx}(\cos x) + \cos x \frac{dy}{dx} e^x$$

$$\frac{dy}{dx} = -e^x \sin x + e^x \cos x \quad \left[\because \frac{d}{dx}(\cos x) = -\sin x \text{ \& \ } \frac{d}{dx} e^x = e^x \right]$$

Again differentiating with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (-e^x \sin x + e^x \cos x)$$

$$= \frac{d}{dx} (-e^x \sin x) + \frac{d}{dx} (e^x \cos x)$$

Again using the product rule

$$\frac{d^2 y}{dx^2} = -e^x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx} e^x + e^x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(e^x)$$

$$\frac{d^2 y}{dx^2} = -e^x \cos x - e^x \sin x - e^x \sin x + e^x \cos x$$

$$[\because \frac{d}{dx}(\cos x) = -\sin x, \frac{d}{dx} e^x = e^x]$$

$$\frac{d^2 y}{dx^2} = -2e^x \sin x \quad [\because -\sin x = \cos(x + \pi/2)]$$

$$\frac{d^2 y}{dx^2} = -2e^x \cos(x + \frac{\pi}{2})$$

Hence proved.

11. If $x = a \cos \theta$ & $y = b \sin \theta$, show that $\frac{d^2 y}{dx^2} = -\frac{b^4}{a^2 y^3}$.

Solution:

Given,

$$x = a \cos \theta \text{equation 1}$$

$$y = b \sin \theta \text{equation 2}$$

If $y = f(\theta)$ and $x = g(\theta)$ that is y is a function of θ and x is also some other

function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

We can write $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Now we have to prove $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a \cos \theta = -a \sin \theta \dots \text{equation 3}$$

$$\text{Similarly, } \frac{dy}{d\theta} = b \cos \theta \dots \text{equation 4}$$

$$\therefore \frac{d}{dx} \cos x = -\sin x \tan x, \frac{d}{dx} \sin x = \cos x$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \cos \theta}{-a \sin \theta} = -\frac{b}{a} \cot \theta$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(-\frac{b}{a} \cot \theta \right)$$

By using chain rule, we get

$$\frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 \theta \frac{d\theta}{dx} \dots \text{equation 5}$$

From equation 3

$$\frac{dx}{d\theta} = -a \sin \theta$$

$$\therefore \frac{d\theta}{dx} = \frac{-1}{a \sin \theta}$$

Putting the value in equation 5

$$\frac{d^2y}{dx^2} = -\frac{b}{a} \operatorname{cosec}^2 \theta \frac{1}{a \sin \theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2 \sin^3 \theta}$$

From equation 1:

$$y = b \sin \theta$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-b}{\frac{a^2 y^3}{b^2}} = -\frac{b^4}{a^2 y^3}$$

Hence proved.

12. If $x = a(1 - \cos^3 \theta)$, $y = b \sin^3 \theta$, prove that $\frac{d^2y}{dx^2} = \frac{32}{27a}$ at $\theta = \frac{\pi}{6}$.

Solution:

Given,

$$x = a(1 - \cos^3 \theta) \dots \dots \text{equation 1}$$

$$y = b \sin^3 \theta \dots \dots \text{equation 2}$$

If $y = f(\theta)$ and $x = g(\theta)$ that is y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\text{We can write } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\text{Let's find } \frac{d^2y}{dx^2}$$

$$\text{We know } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{dx}{d\theta} = 3a \cos^2 \theta \sin \theta$$

$$\therefore \frac{d\theta}{dx} = \frac{1}{3a \cos^2 \theta \sin \theta}$$

Putting the value in equation 5

$$\frac{d^2y}{dx^2} = \sec^2 \theta \frac{1}{3a \cos^2 \theta \sin \theta}$$

Let's find $\frac{d^2y}{dx^2}$

$$\text{We know } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

Now by using chain rule,

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(1 - \cos^2 \theta) = 3a \cos^2 \theta \sin \theta \quad \dots \text{equation 3}$$

Similarly,

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a \sin^2 \theta = 3a \sin^2 \theta \cos \theta \quad \dots \text{equation 4}$$

$$\left[\because \frac{d}{dx} \cos x = -\sin x \text{ \& \& } \frac{d}{dx} \sin x = \cos x \right]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta \sin \theta} = \tan \theta$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\tan \theta)$$

$$\frac{d^2y}{dx^2} = \sec^2 \theta \frac{d\theta}{dx} \quad \dots \text{Equation 5}$$

From equation 3

$$\frac{d^2y}{dx^2} = \frac{1}{3 \operatorname{acos}^4 \theta \sin \theta}$$

Put $\theta = \pi/6$

$$\left(\frac{d^2y}{dx^2}\right) \text{ at } \left(x = \frac{\pi}{6}\right) = \frac{1}{3 \operatorname{acos}^4 \frac{\pi}{6} \sin \frac{\pi}{6}} = \frac{1}{3a \left(\frac{\sqrt{3}}{2}\right)^4 \frac{1}{2}}$$

$$\therefore \left(\frac{d^2y}{dx^2}\right) \text{ at } \left(x = \frac{\pi}{6}\right) = \frac{32}{27a}$$

Hence proved

13. If $x = a(\theta + \sin \theta)$, $y = a(1 + \cos \theta)$, prove that $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$.

Solution:

Given,

$$x = a(\theta + \sin \theta) \dots \text{equation 1}$$

$$y = a(1 + \cos \theta) \dots \text{equation 2}$$

If $y = f(\theta)$ and $x = g(\theta)$ that is y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\text{We can write } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx}\right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(\theta + \sin \theta) = a(1 + \cos \theta) = y$$

Similarly,

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(1 + \cos \theta) = -a \sin \theta \quad \dots \text{equation 4}$$

$$\left[\because \frac{d}{dx} \cos x = -\sin x, \frac{d}{dx} \sin x = \cos x \right]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a \sin \theta}{a(1 + \cos \theta)} = \frac{-\sin \theta}{(1 + \cos \theta)} = \frac{-\sin \theta}{y}$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = -a \frac{d}{dx} \left(\frac{\sin \theta}{y} \right)$$

Using product rule and chain rule together, we get

$$\frac{d^2y}{dx^2} = -a \left(\frac{\sin \theta}{-y^2} \frac{dy}{dx} + \frac{1}{y} \cos \theta \frac{d\theta}{dx} \right)$$

By using equation 3 and 5,

$$\frac{d^2y}{dx^2} = -a \left(\frac{\sin \theta (-a \sin \theta)}{-y^2} + \frac{1}{y} \cos \theta \frac{1}{y} \right)$$

$$\frac{d^2y}{dx^2} = -a \left(\frac{a \sin^2 \theta}{y^2} + \frac{1}{y^2} \cos \theta \right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2} \left(\frac{a \sin^2 \theta}{a(1 + \cos \theta)} + \cos \theta \right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2} \left(\frac{1 - \cos^2 \theta}{(1 + \cos \theta)} + \cos \theta \right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2} \left(\frac{(1 - \cos \theta)(1 + \cos \theta)}{(1 + \cos \theta)} + \cos \theta \right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2} (1 - \cos \theta + \cos \theta)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{a}{y^2}$$

Hence proved.

14. If $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$, find $\frac{d^2y}{dx^2}$.

Given,

$$x = a(\theta - \sin \theta) \text{equation 1}$$

$$y = a(1 + \cos \theta) \text{equation 2}$$

if $y = f(\theta)$ and $x = g(\theta)$ that is y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

We can write

Now we have to find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(\theta - \sin \theta) = a(1 - \cos \theta) \text{equation 3}$$

Similarly,

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(1 + \cos \theta) = -a \sin \theta \text{equation 4}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a \sin \theta}{a(1 - \cos \theta)} = \frac{-\sin \theta}{(1 - \cos \theta)} \text{ Equation 5}$$

Differentiating again with respect to x , we get

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{\sin \theta}{1 - \cos \theta} \right)$$

Using product rule and chain rule together, we get

Solution:

$$\frac{d^2y}{dx^2} = \left[-\frac{1}{1-\cos\theta} \frac{d}{d\theta} \sin\theta - \sin\theta \frac{d}{d\theta} \frac{1}{1-\cos\theta} \right] \frac{d\theta}{dx}$$

Apply chain rule to determine $\frac{d}{d\theta} \frac{1}{1-\cos\theta}$

$$\frac{d^2y}{dx^2} = \left\{ \frac{-\cos\theta}{1-\cos\theta} + \frac{\sin^2\theta}{(1-\cos\theta)^2} \right\} \frac{1}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2} = \left\{ \frac{-\cos\theta(1-\cos\theta) + \sin^2\theta}{(1-\cos\theta)^2} \right\} \frac{1}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2} = \left\{ \frac{-\cos\theta + \cos^2\theta + \sin^2\theta}{(1-\cos\theta)^2} \right\} \frac{1}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2} = \left\{ \frac{1-\cos\theta}{(1-\cos\theta)^2} \right\} \frac{1}{a(1-\cos\theta)} \quad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$\frac{d^2y}{dx^2} = \frac{1}{a(1-\cos\theta)^2}$$

We know $1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$$\frac{d^2y}{dx^2} = \frac{1}{a(2\sin^2\frac{\theta}{2})^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{4a} \operatorname{cosec}^4 \frac{\theta}{2}$$

15. If $x = a(1 - \cos \theta)$, $y = a(\theta + \sin \theta)$, prove that $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

Solution:

Given,

$$y = a(\theta + \sin \theta) \dots \text{equation 1}$$

$$x = a(1 - \cos \theta) \dots \text{equation 2}$$

If $y = f(\theta)$ and $x = g(\theta)$ that is y is a function of θ and x is also some other function of θ .

Using product rule and chain rule together we get

$$\frac{d^2y}{dx^2} = \left\{ \operatorname{cosec} \theta \frac{d}{d\theta} (1 + \cos \theta) + (1 + \cos \theta) \frac{d}{d\theta} \operatorname{cosec} \theta \right\} \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \left\{ \operatorname{cosec} \theta (-\sin \theta) + (1 + \cos \theta) (-\operatorname{cosec} \theta \cot \theta) \right\} \frac{1}{a \sin \theta}$$

$$\frac{d^2y}{dx^2} = \left[-1 - \operatorname{cosec} \theta \cot \theta - \cot^2 \theta \right] \frac{1}{a \sin \theta}$$

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\text{We can write } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Now we have to prove $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(\theta + \sin \theta) = a(1 + \cos \theta) \quad \dots \text{equation 3}$$

Similarly,

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(1 - \cos \theta) = a \sin \theta \quad \dots \text{equation 4}$$

$$\left[\because \frac{d}{dx} \cos x = -\sin x, \frac{d}{dx} \sin x = \cos x \right]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1 + \cos \theta)}{a \sin \theta} = \frac{(1 + \cos \theta)}{\sin \theta} \quad \dots \text{equation 5}$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{(1 + \cos \theta)}{\sin \theta} \right) = \frac{d}{dx} (1 + \cos \theta) \operatorname{cosec} \theta$$

As we have to find $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$

\therefore put $\theta = \pi/2$ in above equation:

$$\frac{d^2y}{dx^2} = \left\{ -1 - \operatorname{cosec} \frac{\pi}{2} \cot \frac{\pi}{2} - \cot^2 \frac{\pi}{2} \right\} \frac{1}{a \sin \frac{\pi}{2}}$$

$$= \frac{(-1 - 0 - 0)}{a}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{a}$$

16. If $x = a(1 - \cos \theta)$, $y = a(\theta + \sin \theta)$, prove that $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

Solution:

Given,

$$y = a(\theta + \sin \theta) \dots \text{equation 1}$$

$$x = a(1 + \cos \theta) \dots \text{equation 2}$$

If $y = f(\theta)$ and $x = g(\theta)$ that is y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\text{We can write: } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Given,

$$y = a(\theta + \sin \theta) \dots \text{equation 1}$$

$$x = a(1 + \cos \theta) \dots \text{equation 2}$$

Now we have to prove $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

Let's find $\frac{d^2y}{dx^2}$

As we have to find $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$

\therefore put $\theta = \pi/2$ in above equation:

$$\frac{d^2y}{dx^2} = \left[-1 - \operatorname{cosec} \frac{\pi}{2} \cot \frac{\pi}{2} - \cot^2 \frac{\pi}{2}\right] \frac{1}{a \sin \frac{\pi}{2}} = \frac{(-1 - 0 - 0)}{a}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{a}$$

We know, $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(1 + \sin \theta) = a(1 + \cos \theta) \dots \text{equation 3}$$

Similarly,

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(1 + \cos \theta) = -a \sin \theta \dots \text{equation 4}$$

$$\left[\because \frac{d}{dx} \cos x = -\sin x, \frac{d}{dx} \sin x = \cos x \right]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1 + \cos \theta)}{-a \sin \theta} = -\frac{(1 + \cos \theta)}{\sin \theta} \dots \text{equation 5}$$

Differentiating again with respect to x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(-\frac{(1 + \cos \theta)}{\sin \theta} \right) = -\frac{d}{dx} (1 + \cos \theta) \operatorname{cosec} \theta$$

Using product rule and chain rule together

$$\frac{d^2y}{dx^2} = -\left[\operatorname{cosec} \theta \frac{d}{d\theta} (1 + \cos \theta) + (1 + \cos \theta) \frac{d}{d\theta} \operatorname{cosec} \theta \right] \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = -\left[\operatorname{cosec} \theta (-\sin \theta) + (1 + \cos \theta) (-\operatorname{cosec} \theta \cot \theta) \right] \frac{1}{(-a \sin \theta)}$$

$$\frac{d^2y}{dx^2} = \left[-1 - \operatorname{cosec} \theta \cot \theta - \cot^2 \theta \right] \frac{1}{a \sin \theta}$$

17. If $x = \cos \theta$, $y = \sin^3 \theta$, prove that $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3 \sin^2 \theta (5 \cos^2 \theta - 1)$.

Given,

$$y = \sin^3 \theta \text{equation 1}$$

$$x = \cos \theta \text{equation 2}$$

If $y = f(\theta)$ and $x = g(\theta)$, that is y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

We can write: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

To prove: $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3 \sin^2 \theta (5 \cos^2 \theta - 1)$

Now we have to find $\frac{d^2y}{dx^2}$

We know, $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx}\right)$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = -\sin \theta \text{equation 3}$$

Applying chain rule to differentiate $\sin^3 \theta$, then

$$\frac{dy}{d\theta} = 3 \sin^2 \theta \cos \theta \text{equation 4}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3 \sin^2 \theta \cos \theta}{-\sin \theta} = -3 \sin \theta \cos \theta \text{equation 5}$$

Again differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx}\right)$$

Solution:

$$\frac{d^2y}{dx^2} - \frac{d}{dx}(-3 \sin \theta \cos \theta)$$

Applying product rule and chain rule together, we get

$$\frac{d^2y}{dx^2} - 3\left[\sin \theta \frac{d}{d\theta} \cos \theta + \cos \theta \frac{d}{d\theta} \sin \theta\right] \frac{d\theta}{dx}$$

Put the value of $d\theta/dx$

$$\frac{d^2y}{dx^2} - 3\{-\sin^2 \theta + \cos^2 \theta\} \frac{1}{\sin \theta}$$

Multiplying y both sides to approach towards the expression we want to prove

$$y \frac{d^2y}{dx^2} - 3\{-\sin^2 \theta + \cos^2 \theta\} \frac{y}{\sin \theta}$$

Substitute the value of y

$$y \frac{d^2y}{dx^2} = 3\{-\sin^2 \theta + \cos^2 \theta\} \sin^2 \theta$$

Adding equation 5 and squaring we get

$$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\{-\sin^2 \theta + \cos^2 \theta\} \sin^2 \theta + 9 \sin^2 \theta \cos^2 \theta$$

$$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3 \sin^2 \theta [-\sin^2 \theta + \cos^2 \theta + 3 \cos^2 \theta]$$

$$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3 \sin^2 \theta [5 \cos^2 \theta - 1]$$

18. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \cdot \frac{dy}{dx} + y \cos^2 x = 0$.

Solution:

Given, $y = \sin(\sin x)$ equation 1

To prove: $\frac{d^2y}{dx^2} + \tan x \cdot \frac{dy}{dx} + y \cos^2 x = 0$

Now we have to find $\frac{d^2y}{dx^2}$

We know $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So, first we have to find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx} \sin(\sin x)$$

Using chain rule, we will differentiate the above expression

$$\text{Let } t = \sin x \Rightarrow \frac{dt}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$\frac{dy}{dx} = \cos t \cos x = \cos(\sin x) \cos x \quad \dots \text{equation 2}$$

Again differentiating with respect to x applying product rule, we get

$$\frac{d^2y}{dx^2} = \cos x \frac{d}{dx} \cos(\sin x) + \cos(\sin x) \frac{d}{dx} \cos x$$

Using chain rule we get

$$\frac{d^2y}{dx^2} = -\cos x \cos x \sin(\sin x) - \sin x \cos(\sin x)$$

$$\frac{d^2y}{dx^2} = -y \cos^2 x - \tan x \cos x \cos(\sin x)$$

And using equation 2, we have:

$$\frac{d^2y}{dx^2} = -y \cos^2 x - \tan x \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} + y \cos^2 x + \tan x \frac{dy}{dx} = 0$$