NCERT Solutions for Class-XII Maths

Chapter-11.2

NCERT Maths Class 12

1. Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

1. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$

If two lines are perpendicular, then the angle between the two is $\theta = 90°$

$\Rightarrow$ For perpendicular lines, $|l_1l_2 + m_1m_2 + n_1n_2| = \cos 90° = 0$, i.e.

$|l_1l_2 + m_1m_2 + n_1n_2| = 0$

So, in order to check if the three lines are mutually perpendicular, we compute $|l_1l_2 + m_1m_2 + n_1n_2|$ for all the pairs of the three lines.

Now let the direction cosines of L_1, L_2 and L_3 be l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3.

First, consider

$|l_1l_2 + m_1m_2 + n_1n_2| = \left|\left(\frac{12}{13} \times \frac{4}{13}\right) + \left(\frac{-3}{13} \times \frac{12}{13}\right) + \left(\frac{-4}{13} \times \frac{3}{13}\right)\right| = \frac{48}{13} + \left(\frac{-36}{13}\right) + \left(\frac{-12}{13}\right) = \frac{48 + (-48)}{13} = 0$

$\Rightarrow L_1 \perp L_2$

Next, consider ….(i)

$|l_2l_3 + m_2m_3 + n_2n_3| = \left|\left(\frac{4}{13} \times \frac{3}{13}\right) + \left(\frac{12}{13} \times \frac{-4}{13}\right) + \left(\frac{3}{13} \times \frac{12}{13}\right)\right| = \frac{12}{13} + \left(\frac{-48}{13}\right) + \frac{36}{13} = \frac{(-48) + 48}{13} = 0$

$\Rightarrow L_2 \perp L_3$ …..(ii)

Now consider

$|l_3l_1 + m_3m_1 + n_3n_1| = \left|\left(\frac{3}{13} \times \frac{12}{13}\right) + \left(\frac{-4}{13} \times \frac{-3}{13}\right) + \left(\frac{12}{13} \times \frac{-4}{13}\right)\right| = \frac{36}{13} + \frac{12}{13} + \left(\frac{-48}{13}\right) = \frac{48 + (-48)}{13} = 0$

$\Rightarrow L_1 \perp L_3$ …..(iii)

$\therefore$ By (i), (ii) and (iii), we have

L_1, L_2 and L_3 are mutually perpendicular.

2. Show that the line through the points (1, −1, 2) (3, 4, −2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

2. We know that

Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular if the angle between them is θ = 90°, i.e. $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Also, we know that the direction ratios of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is taken as $x_2 - x_1, y_2 - y_1, z_2 - z_1$ (or $x_1 - x_2, y_1 - y_2, z_1 - z_2$).

⇒ The direction ratios of the line through the points (1, – 1, 2) and (3, 4, – 2) is:

$a_1 = 3 - 1 = 2$, $b_1 = 4 - (-1) = 4 + 1 = 5$, $c_1 = -2 - 2 = -4$

and the direction ratios of the line through the points (0, 3, 2) and (3, 5, 6) is:

$a_2 = 3 - 0 = 3$, $b_2 = 5 - 3 = 2$, $c_2 = 6 - 2 = 4$

Now, consider

$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 5 \times 2 + (-4) \times 4 = 6 + 10 + (-16) = 16 + (-16) = 0$

⇒ The line through the points (1, – 1, 2), (3, 4, – 2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

3. Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (−1, −2, 1), (1, 2, 5).

3. Let AB be the line through the points, (4, 7, 8) and (2, 3, 4), and CD be the line through the points (−1, −2, 1) and (1, 2, 5).

The directions ratios, a_1, b_1, c_1, of AB are (2 − 4), (3 − 7), and (4 − 8) i.e., −2, −4, and −4.

The direction ratios, a_2, b_2, c_2, of CD are (1 − (−1)), (2 − (−2)), and (5 − 1) i.e., 2, 4, and 4.

AB will be parallel to CD, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, AB is parallel to CD.

4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

4. Vector equation of a line that passes through a given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + l\,\vec{b}$.

So, here the position vector of the point (1, 2, 3) is given by $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and the parallel vector is $3\hat{i} + 2\hat{j} - 2\hat{k}$.

$\therefore$ The vector equation of the required line is:

$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + l\,(3\hat{i} + 2\hat{j} - 2\hat{k})$, where l is a real number.

5. Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

5. It is given that the line passes through the point with position vector

$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ ….(1)

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ …..(2)

It is known that a line through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is given by

the equation, $\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$

This is the required equation of the line in vector form.

$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$

$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$

Eliminating λ, we obtain the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

6. Find the Cartesian equation of the line which passes through the point (−2, 4, −5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

6. The Cartesian equation of a line through a point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

Here, The point (x_1, y_1, z_1) is (–2, 4, –5) and the direction ratios are :

a = 3, b = 5, c = 6

$\Rightarrow$ The Cartesian equation of the required line is:

$$\frac{x-(-2)}{3} = \frac{y-4}{5} = \frac{z-(-5)}{6} \text{ fi } \frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$$

7. The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its vector form.

7. The Cartesian equation of the line is

$\frac{x - 5}{3}=\frac{y+4}{7}=\frac{z - 6}{2}$ …..(1)

The given line passes through the point (5, −4, 6). The position vector of this point is

$\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector, $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

It is known that the line through position vector $\vec{a}$ and in the direction of the vector $\vec{b}$ is given by the equation, $\vec{r}=\vec{a}+l\,\vec{b}, l$ ŒR

fi $\vec{r}=(5\hat{i}-4\hat{j}+6\hat{k})+l\,(3\hat{i}+7\hat{j}+2\hat{k})$

This is the required equation of the given line in vector form.

8. Find the vector and the Cartesian equations of the lines that pass through the origin and (5, −2, 3).

8. We know that

The vector equation of as line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+l\,(\vec{b}-\vec{a})$.

Here, the position vectors of the two points (0, 0, 0) and (5, –2, 3) are $\vec{a}=0\hat{i}+0\hat{j}+0\hat{k}$ and $\vec{b}=5\hat{i}-2\hat{j}+3\hat{k}$ respectively.

So, The vector equation of the required line is:

$\vec{r}=0\hat{i}+0\hat{j}+0\hat{k}+l\,[(5\hat{i}-2\hat{j}+3\hat{k})-(0\hat{i}+0\hat{j}+0\hat{k})]$

$\Rightarrow \vec{r}=l\,(5\hat{i}-2\hat{j}+3\hat{k})$

Now, we know that

Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

So, the Cartesian equation of the line that passes through the origin (0, 0, 0) and (5, –2, 3) is

$\frac{x - 0}{5 - 0}=\frac{y - 0}{-2 - 0}=\frac{z - 0}{3 - 0}$ fi $\frac{x}{5}=\frac{y}{-2}=\frac{z}{3}$

9. Find the vector and the Cartesian equations of the line that passes through the points (3, −2, −5), (3, −2, 6).

9. Let the line passing through the points, P (3, −2, −5) and Q (3, −2, 6), be PQ.

Since PQ passes through P (3, −2, −5), its position vector is given by,

$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$

The direction ratios of PQ are given by,

(3 − 3) = 0, (−2 + 2) = 0, (6 + 5) = 11

The equation of the vector in the direction of PQ is

$\vec{b} = 0.\hat{i} - 0.\hat{j} + 11\hat{k} = 11\hat{k}$

The equation of PQ in vector form is given by,

$\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$ -

$\Rightarrow \vec{r} = (3\hat{i} - 2\hat{j} - 5\hat{k}) + 11\lambda\hat{k}$ -

The equation of PQ in Cartesian form is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

$$\frac{x - 3}{0} = \frac{y + 2}{0} = \frac{z + 5}{11}$$

10. Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$ -and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

10. We know that

If θ is the acute angle between $\vec{r} = \vec{a}_1 + l\,\vec{b}_1$ and $\vec{r} = \vec{a}_2 + m\vec{b}_2$ then

$$\cos q = \left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right| \qquad \text{.....(i)}$$

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + l\,(3\hat{i} + 2\hat{j} + 6\hat{k})$ and $\vec{r} = 7\hat{i} - 6\hat{k} + m(\hat{i} + 2\hat{j} + 2\hat{k})$

Here $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$

So form (i), we have

$$\cos q = \frac{(3\hat{i} + 2\hat{j} + 6\hat{k}).(\hat{i} + 2\hat{j} + 2\hat{k})}{|3\hat{i} + 2\hat{j} + 6\hat{k}|.|\hat{i} + 2\hat{j} + 2\hat{k}|} \qquad \text{.....(ii)}$$

$\because |a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$

\ $|3\hat{i} + 2\hat{j} + 6\hat{k}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

And $|\hat{i} + 2\hat{j} + 2\hat{k}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

Now $\because (a_1\hat{i} + b_1\hat{j} + c_1\hat{k}).(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) a_1a_2 + b_1b_2 + c_1c_2$

\ $(3\hat{i} + 2\hat{j} + 6\hat{k}).(\hat{i} + 2\hat{j} + 2\hat{k}) = 3 \times 1 + 2 \times 2 + 6 \times 2 = 3 + 4 + 12 = 19$

⇒ By (iii), we have

$$\cos\theta = \frac{19}{7\times 3} = \frac{19}{21}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

Here, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$

So, from (i) we have

$$\cos\theta = \left|\frac{(\hat{i} - \hat{j} - 2\hat{k})}{|\hat{i} - \hat{j} - 2\hat{k}||3\hat{i} - 5\hat{j} - 4\hat{k}|}\right| \qquad \ldots\text{(iii)}$$

$\because |a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$

$\therefore |\hat{i} - \hat{j} - 2\hat{k}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6} = \sqrt{3}\times\sqrt{2}$

And $|3\hat{i} - 5\hat{j} - 4\hat{k}| = \sqrt{3^2 + (-5)^2 + (-4)^2} = \sqrt{9+25+16} = \sqrt{50} = 5\sqrt{2}$

Now $(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}).(a_2\hat{i} + b_1\hat{j} + c_1\hat{k}).(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = a_1a_2 + b_1b_2 + c_1c_2$

$\therefore (\hat{i} - \hat{j} - 2\hat{k}).(3\hat{i} - 5\hat{j} - 4\hat{k}) = 1 \times 3 + (-1) \times (-5) + (-2) \times (-4) = 3 + 5 + 8 = 16$

$\Rightarrow$ By (iii), we have

$$\cos\theta = \frac{16}{\sqrt{3}\times\sqrt{2}\times 5\sqrt{2}} = \frac{16}{5\times 2\sqrt{3}} = \frac{8}{5\sqrt{3}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

11. Find the angle between the following pairs of lines:

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

11. (i) Let $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to the pair of lines,

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$$

$\therefore \vec{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{b}_2 = -\hat{i} + 8\hat{j} + 4\hat{k}$

$|\vec{b}_1| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$

$|\vec{b}_2| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$

$\vec{b}_1.\vec{b}_2 = (2\hat{i} + 5\hat{j} - 3\hat{k}).(-\hat{i} + 8\hat{j} + 4\hat{k})$

$= 2(-1) + 5 \times 8 + (-3).4$
$= -2 + 40 - 12$
$= 26$

The angle, Q, between the given pair of lines is given by the relation,

$$\cos Q = \left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

$$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) Let $\vec{b}_1, \vec{b}_2$ be the vectors parallel to the given pair of lines,

$\frac{x-5}{4} = \frac{y-5}{1} = \frac{z-3}{8}$ and $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ respectively.

$\vec{b}_2 = 2\hat{i} + 2\hat{j} + \hat{k}$

$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$

$\therefore |\vec{b}_1| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$

$|\vec{b}_2| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$

$\vec{b}_1.\vec{b}_2 = (2\hat{i} + 2\hat{j} + \hat{k}).(4\hat{i} + \hat{j} + 8\hat{k})$
$= 2 \times 4 + 2 \times 1 + 1 \times 8$
$= 8 + 2 + 8$
$= 18$

If Q is the angle between the given pair of lines, then $\cos Q = \left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$-

$$\Rightarrow \cos Q = \frac{18}{3 \times 9} = \frac{2}{3}$$

$$\Rightarrow Q = \cos^{-1} = \left(\frac{2}{3}\right)$$

12. Find the values of p so the line $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

12. For any two lines to be at right angles, the angle between them should be $\theta = 90°$.
$\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$, where a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines.
The standard form of a pair of Cartesian lines is:

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2} \quad \text{....(i)}$$

Now, first we rewrite the given equations according to the standard form, i.e.

$$\frac{-(x-1)}{3}=\frac{7(y-2)}{2p}=\frac{z-3}{2} \text{ and } \frac{-7(x-1)}{3p}=\frac{y-5}{1}=\frac{-(z-6)}{5}, \text{ i.e.}$$

$$\frac{x-1}{-3}=\frac{y-2}{2p/7}=\frac{z-3}{2} \text{ and } \frac{x-1}{-3p/7}=\frac{y-5}{1}=\frac{z-6}{-5} \quad \text{....(ii)}$$

Now, comparing (i) and (ii), we get

$$a_1=-3, b_1=\frac{2p}{7}, c_1=2 \text{ and } a_2=\frac{-3p}{7}, b_2=1, c_2=-5$$

Now, as both the lines are at right angles,

so $a_1a_2+b_1b_2+c_1c_2=0$

$$\Rightarrow (-3)\times\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)\times 1+2\times(-5)=0$$

$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}-10=0$$

$$\Rightarrow \frac{9p+2p}{7}=10$$

$$\Rightarrow \frac{11p}{7}=10$$

$$\Rightarrow 11p=70$$

$$\Rightarrow p=\frac{70}{11}$$

13. Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

13. The equations of the given lines are $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

The direction ratios of the given lines are 7, −5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a1, b1, c1 and a2, b2, c2, are perpendicular to each other, if $a_1a_2+b_1b_2+c_1c_2=0$

$\therefore\ 7\times 1+(-5)\times 2+1\times 3$

$=7-10+3$

$=0$

Therefore, the given lines are perpendicular to each other.

14. Find the shortest distance between the lines

$\vec{r}=(\hat{i}+2\hat{j}+\hat{k})$ and $+\lambda(\hat{i}-\hat{j}+\hat{k})$ and

$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu(2\hat{i}+\hat{j}+2\hat{k})$

14. We know that

Shortest distance between two lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| \quad \text{....(i)}$$

Here $\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}, \vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$ and

$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}, \vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$

Now $(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) - (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$

$\therefore \ \vec{a}_2 - \vec{a}_1 = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 2\hat{k}$ ....(ii)

Now, $\vec{b}_1 \times \vec{b}_2 = (\hat{i} - \hat{j} + \hat{k}) \times (2\hat{i} + \hat{j} + 2\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -3\hat{i} + 3\hat{k}$$

$\Rightarrow \ \vec{b}_1 \times \vec{b}_2 = -3\hat{i} + 3\hat{k}$ ....(iii)

$\Rightarrow \ |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + 3^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$ .....(iv)

Now, $\because (a_1\hat{i} + b_1\hat{j} + c_1\hat{k}).(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = a_1a_2 + b_1b_2 + c_1c_2$

$\therefore \ (\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1) = (-3\hat{i} + 3\hat{k}).(\hat{i} - 3\hat{j} - 2\hat{k}) = -3 - 6 = -9$ ....(v)

Now, using (i), we have

The shortest distance between the two lines, $d = \left| \frac{-9}{3\sqrt{2}} \right| = \frac{9}{3\sqrt{2}}$ [From (iv) and (v)]

$$= \frac{3}{\sqrt{2}}$$

Rationalizing the fraction by multiplying the numerator and denominator by $\sqrt{2}$

$\Rightarrow \ d = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

15. Find the shortest distance between the lines

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

15. The given lines are

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is given by,

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}} \quad \text{....(1)}$$

Comparing the given equations, we obtain

$x_1 = -1,\ y_1 = -1,\ z_1 = -1$

$a_1 = 7,\ b_1 = -6,\ c_1 = 1$

$x_2 = 3,\ y_2 = 5,\ z_2 = 7$

$a_2 = 1,\ b_2 = -2,\ c_2 = 1$

Then, Then, $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

$= 4(-6+2) - 6(7-1) + 8(-14+6)$

$= -16 - 36 - 64$

$= -116$

$\Rightarrow \sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2} = \sqrt{(-6+2)^2(1+7)^2+(-14+6)^2}$

$= \sqrt{16+36+64}$

$= \sqrt{116}$

$2\sqrt{29}$

Substituting all the values in equation (1), we obtain

$$d=\frac{-116}{2\sqrt{29}}=\frac{-58}{\sqrt{29}}=\frac{-2\times 29}{\sqrt{29}}=-2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is $2\sqrt{29}$ units.

16. Find the shortest distance between the lines whose vector equations are

$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}+3\hat{j}+2\hat{k}\right)$ and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu(2\hat{i}+3\hat{j}+\hat{k})$

16. We know that

Shortest distance between two lines $\vec{r}=\vec{a}_1+l\,b_1$ and $\vec{r}=\vec{a}_2+m\vec{b}_2$ is

$$d=\left|\frac{\left(\vec{b}_1 \text{¥} \vec{b}_2\right).\left(\vec{a}_1 \text{¥} \vec{a}_2\right)}{\left|\vec{b}_1 \text{¥} \vec{b}_2\right|}\right|$$

Here, $\vec{a}_1=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}_1=\hat{i}-3\hat{j}+2\hat{k}$ and

$\vec{a}_2=4\hat{i}+5\hat{j}+6\hat{k}$, $\vec{b}_2=2\hat{i}+3\hat{j}+\hat{k}$

Now, $\because \left(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right)-\left(x_2\hat{i}+y_2\hat{j}+z_2\hat{k}\right)=(x_1-x_2)\hat{i}+(y_1-y_2)\hat{j}+(z_1-z_2)\hat{k}$

$\therefore\ \vec{a}_2-\vec{a}_1=\left(4\hat{i}+5\hat{j}+6\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=\left(3\hat{i}+3\hat{j}+3\hat{k}\right)$

Now, $\vec{b}_1\times\vec{b}_2=\left(\hat{i}-3\hat{j}+2\hat{k}\right)\times\left(2\hat{i}+3\hat{j}+\hat{k}\right)$

$$=\begin{vmatrix} i & j & k \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}=-9\hat{i}+3\hat{j}+9\hat{k}$$

$\Rightarrow\ \vec{b}_1\times\vec{b}_2=-9\hat{i}+3\hat{j}+9\hat{k}$ ….(iii)

$\Rightarrow\ \left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(-9)^2+3^2+9^2}=\sqrt{81+9+81}=\sqrt{171}=3\sqrt{19}$ ….(iv)

Now, $\because \left(a_1\hat{i}+b_1\hat{j}+c_1\hat{k}\right).\left(a_2\hat{i}+b_2\hat{j}+c_2\hat{k}\right)=a_1a_2+b_1b_2+c_1c_2$

$\because \left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_2\times\vec{a}_1\right)=\left(-9\hat{i}+3\hat{j}+9\hat{k}\right).\left(3\hat{i}+3\hat{j}+3\hat{k}\right)=-27+9+27=9$ …(v)

Now, using (i), we have

The shortest distance between the two lines, d

$$=\left|\frac{9}{3\sqrt{19}}\right|=\frac{9}{3\sqrt{19}}=\frac{3}{\sqrt{19}}$$

17. Find the shortest distance between the lines whose vector equations are

$\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and

$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$

17. The given lines are

$\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$

$\Rightarrow \vec{r}=(\hat{i}-2\hat{j}+3\hat{k})+t(-\hat{i}+\hat{j}-2\hat{k})$ ….(1)

$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$

$\Rightarrow \vec{r}=(\hat{i}-\hat{j}+\hat{k})+s(\hat{i}+2\hat{j}-2\hat{k})$ …..(2)

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right).(\vec{a}_1-\vec{a}_2)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \quad ….(3)$$

For the given equations,

$\vec{a}_1=\hat{i}-2\hat{j}+3\hat{k}$

$\vec{b}_1=-\hat{i}+\hat{j}-2\hat{k}$

$\vec{a}_2=\hat{i}-\hat{j}-\hat{k}$

$\vec{b}_2=\hat{i}+2\hat{j}-2\hat{k}$

$\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} i & j & k \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{4+16+9} = \sqrt{29}$

$\therefore \left(\vec{b}_1 \times \vec{b}_2\right).\left(\vec{a}_2 - \vec{a}_1\right) = \left(2\hat{i} - 4\hat{j} - 3\hat{k}\right).\left(\hat{j} - 4\hat{k}\right) = -4 + 12 = 8$

Substituting all the values in equation (3), we obtain

$$d = \left|\frac{8}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the liens is $\frac{8}{\sqrt{29}}$ units.