EXERCISE 9.1

1. Test the continuity of the following function at the origin:

$$f(x)=\begin{cases}\frac{x}{|x|}, x\neq 0\\ 1, x=0\end{cases}$$

Solution:

Given

$$f(x)=\begin{cases}\frac{x}{|x|}, x\neq 0\\ 1, x=0\end{cases}$$

Consider LHL at x = 0

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$$

$$\lim_{h\to 0}\frac{-h}{|-h|}=\lim_{h\to 0}\frac{-h}{h}=\lim_{h\to 0} -1=-1$$

Consider RHL at x = 0

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$$

$$\lim_{h\to 0}\frac{h}{|h|}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0} 1=1$$

$$\therefore \lim_{x\to 0^+} f(x)\neq \lim_{x\to 0^-} f(x)$$

Hence LHL ≠ RHL

Hence f(x) is discontinuous at origin.

2. A function f(x) is defined as

$$f(x)=\begin{cases}\frac{x^2-x-6}{x-3},\ if\ x\neq 3\\ 5,\ if\ x=3\end{cases}$$

Show that f(x) is continuous at x = 3.

Solution:

Given

$$f(x)=\begin{cases}\frac{x^2-x-6}{x-3},\ if\ x\neq 3\\ 5,\ if\ x=3\end{cases}$$

Consider LHL at x = 3

$$\lim_{x\to 3^-} f(x)=\lim_{h\to 0} f(3-h)$$

$$\lim_{h\to 0}\frac{(3-h)^2-(3-h)-6}{(3-h)-3}=\lim_{h\to 0}\frac{9+h^2-6h-3+h-6}{-h}=\lim_{h\to 0}\frac{h^2-5h}{-h}=\lim_{h\to 0}(5-h)=5$$

Consider RHL at x = 3

$$\lim_{x\to 3^+} f(x)=\lim_{h\to 0} f(3+h)$$

$$\lim_{h\to 0}\frac{(3+h)^2-(3+h)-6}{(3+h)-3}=\lim_{h\to 0}\frac{9+h^2+6h-3-h-6}{h}=\lim_{h\to 0}\frac{h^2+5h}{h}=\lim_{h\to 0}(5+h)=5$$

Now, f (3) = 5

$$\therefore \lim_{x\to 3^+} f(x)=\lim_{x\to 3^-} f(x)=f(3)$$

Hence f(x) is continuous at x = 3

3. A function f(x) is defined as

$$f(x)=\begin{cases}\frac{x^2-9}{x-3},\ if\ x\neq 3\\ 6,\ if\ x=3\end{cases}$$

Show that f(x) is continuous at x = 3.

Solution:

Given

$$f(x)=\begin{cases}\frac{x^2-9}{x-3};\ if\ x\neq 3\\ 6;\ if\ x=3\end{cases}$$

Consider LHL at x = 3

$$\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3-h)$$

$$\lim_{h \to 0} \frac{(3-h)^2 - 9}{(3-h) - 3} = \lim_{h \to 0} \frac{3^2 + h^2 - 6h - 9}{3 - h - 3} = \lim_{h \to 0} \frac{h^2 - 6h}{-h} == \lim_{h \to 0} \frac{h(h-6)}{-h} = \lim_{h \to 0} (6-h) = 6$$

Consider RHL at x = 3

$$= \lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h)$$

$$\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h)$$

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{3 + h - 3} = \lim_{h \to 0} \frac{3^2 + h^2 + 6h - 9}{h} = \lim_{h \to 0} \frac{h^2 + 6h}{h} = \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} (6+h) = 6$$

We have f (3) = 6

$$\therefore \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

Hence f(x) is continuous at x = 3

4. $f(x) = \begin{cases} \frac{x^2-1}{x-1};\ if\ x \neq 1 \\ 2;\ if\ x = 1 \end{cases}$

Find whether f(x) is continuous at x = 1.

Solution:

Given

$$f(x) = \begin{cases} \frac{x^2-1}{x-1};\ if\ x \neq 1 \\ 2;\ if\ x = 1 \end{cases}$$

Consider LHL at x = 1

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$$

$$\lim_{h \to 0} \frac{(1-h)^2 - 1}{(1-h) - 1} = \lim_{h \to 0} \frac{1 + h^2 - 2h - 1}{1 - h - 1} = \lim_{h \to 0} \frac{h^2 - 2h}{-h} = \lim_{h \to 0} \frac{h(h-2)}{-h} = \lim_{h \to 0} (2-h) = 2$$

Consider RHL at x = 1

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$\lim_{h\to 0} \frac{(1+h)^2-1}{(1+h)-1} = \lim_{h\to 0} \frac{1+h^2+2h-1}{1+h-1} = \lim_{h\to 0} \frac{h^2+2h}{h} = \lim_{h\to 0} \frac{h(h+2)}{h} = \lim_{h\to 0}(2+h) = 2$$

Given f (1) = 2

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

Hence f(x) is continuous at x = 1

5. $If\ f(x) = \begin{cases} \frac{sin3x}{x};\ when\ x \neq 0 \\ 1;\ when\ x = 0 \end{cases}$

Find whether f(x) is continuous at x = 0.

Solution:

Given

$$f(x) = \begin{cases} \frac{\sin 3x}{x};\ when\ x \neq 0 \\ 1;\ when\ x = 0 \end{cases}$$

Consider LHL at x = 0

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$\lim_{h\to 0} \frac{\sin(-3h)}{-h} = \lim_{h\to 0} \frac{-\sin(3h)}{-h} = \lim_{h\to 0} \frac{3\sin(3h)}{3h} = 3\lim_{h\to 0} \frac{\sin(3h)}{3h} = 3\cdot 1 = 3$$

Consider RHL at x = 0

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(h)$$

$$\lim_{h\to 0} \frac{\sin 3h}{h} = \lim_{h\to 0} \frac{3\sin 3h}{3h} = 3\lim_{h\to 0} \frac{\sin(3h)}{3h} = 3\cdot 1 = 3$$

Given f (0) = 1

f (x) to be continuous at x = a

But here,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$$

Hence f (x) is discontinuous at x = 0

6. $$\boldsymbol{If\ f(x) = \begin{cases} e^{\frac{1}{x}};\ when\ x \neq 0 \\ 1;\ when\ x = 0 \end{cases}}$$

Find whether f(x) is continuous at x = 0.

Solution:

Given

$$f(x) = \begin{pmatrix} e^{\frac{1}{x}}, if x \neq 0 \\ 1, if x = 0 \end{pmatrix}$$

Consider LHL at x = 0

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$\lim_{h \to 0} e^{\frac{-1}{h}} = \lim_{h \to 0} \left(\frac{1}{e^{\frac{1}{h}}}\right) = \frac{1}{\lim_{h \to 0} e^{\frac{1}{h}}} = 0$$

Consider RHL at x = 0

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(h)$$

$$\lim_{h \to 0} e^{\frac{1}{h}} = \infty$$

We have f (0) = 1

It is known that for a function f (x) to be continuous at x = a

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

But

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Hence f(x) is discontinuous at x = 0

7. Let $f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & when\ x \neq 0 \\ 1, & when x = 0. \end{cases}$ Show that $f(x)$ is discontinuous at $x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & when\ x \neq 0 \\ 1, & when x = 0. \end{cases}$$

Consider,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{2\sin^2\frac{x}{2}}{x^2}\right)$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{2\sin^2\frac{x}{2}}{4\left(\frac{x^2}{4}\right)}\right)$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{2\left(\sin\frac{x}{2}\right)^2}{4\left(\frac{x}{2}\right)^2}\right)$$

$$\Rightarrow \lim_{x \to 0} f(x) = \frac{2}{4} \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

$$\Rightarrow \lim_{x \to 0} f(x) = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$$

We have f (0) = 1

$$\lim_{x \to 0} f(x) \neq f(0)$$

Thus f (x) is discontinuous at x = 0

8. *Show that* $f(x) = \begin{cases} \frac{x-|x|}{2}, \ when\ x \neq 0 \\ 2, \ when x = 0. \end{cases}$ *is discontinuous at* $x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{x-|x|}{2}, \ when\ x \neq 0 \\ 2, \ when x = 0. \end{cases}$$

The given function can be written as

$$f(x) = \begin{cases} \frac{x-x}{2}, when x > 0 \\ \frac{x+x}{2}, when x < 0 \\ 2, when x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0, when x > 0 \\ x, when x < 0 \\ 2, when x = 0 \end{cases}$$

Consider LHL at x = 0

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h) = 0$$

Consider LHL at x = 0

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h) = 0$$

Consider RHL at x = 0

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$\lim_{h \to 0} 0 = 0$$

And we have f (0) = 2

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$$

Hence, f(x) is discontinuous at x = 0

9. $\textit{Show that } f(x) = \begin{cases} \frac{|x-a|}{x-a}, \ when\ x \neq a \\ 1, \ when x = a. \end{cases} \textit{ is discontinuous at } x = a$

Solution:

Given

$$f(x) = \begin{cases} \frac{|x-a|}{x-a}, \ when\ x \neq a \\ 1, \ when x = a. \end{cases}$$

The given function can be written as

$$f(x) = \begin{cases} \frac{x-a}{x-a}, when x > a \\ \frac{a-x}{x-a}, when x < a \\ 1, when x = a \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 1, when x > a \\ -1, when x < a \\ 1, when x = a \end{cases}$$

$$\Rightarrow f(x) = \begin{pmatrix} 1, when x \geq a \\ -1, when x < a \end{pmatrix}$$

Consider LHL at x = a

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

$$= \lim_{h \to 0} (-1) = -1$$

Consider RHL at x = a

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

$$\lim_{h \to 0} (1) = 1$$

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

Thus f (x) is discontinuous at x = a

10. Discuss the continuity of the following functions at the indicated point(s):

(i) $f(x) = \begin{cases} |x| \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ $at x = 0$

Solution:

Given

$$f(x) = \begin{cases} |x| \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Consider,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| \cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x \to 0} f(x) = \lim_{x \to 0} |x| \lim_{x \to 0} \cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x \to 0} f(x) = 0 \times \lim_{x \to 0} \cos\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \lim_{x \to 0} f(x) = f(0)$$

Hence f(x) is continuous at x = 0

(ii) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} at x = 0$

Solution:

Given

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Consider,

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x \to 0} x^2 \lim_{x \to 0} \sin\left(\frac{1}{x}\right) = 0 \times \lim_{x \to 0} \sin\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \lim_{x \to 0} f(x) = f(0)$$

Hence f(x) is continuous at x = 0

(iii) $f(x) = \begin{cases} (x-a) \sin\left(\frac{1}{x-a}\right), & x \neq a \\ 0, & x = a \end{cases} at x = a$

Solution:

Given

$$f(x) = \begin{cases} (x-a) \sin\left(\frac{1}{x-a}\right), & x \neq a \\ 0, & x = a \end{cases}$$

Now substitute x – a = y in above equation then we get,

$$\lim_{x \to a} (x - a) \sin\left(\frac{1}{x - a}\right) = \lim_{y \to 0} y \sin\left(\frac{1}{y}\right)$$

$$= \lim_{y \to 0} y \lim_{y \to 0} \sin\left(\frac{1}{y}\right) = 0 \times \lim_{y \to 0} \sin\left(\frac{1}{y}\right) = 0$$

$$\Rightarrow \lim_{x \to a} f(x) = f(a) = 0$$

Hence f (x) is continuous at x = a

(iv) $f(x) = \begin{cases} \frac{e^x - 1}{\log(1+2x)}, if & x \neq a \\ 7, if & x = 0 \end{cases} at x = 0$

Solution:
Given

$$f(x) = \begin{cases} \frac{e^x - 1}{\log(1+2x)}, if & x \neq a \\ 7, if & x = 0 \end{cases}$$

Consider,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x - 1}{\log(1 + 2x)}$$

$$\Rightarrow \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x - 1}{\frac{2x \log(1+2x)}{2x}}$$

$$\Rightarrow \lim_{x \to 0} f(x) = \frac{1}{2} \lim_{x \to 0} \frac{\left(\frac{e^x - 1}{x}\right)}{\left(\frac{\log(1+2x)}{2x}\right)}$$

$$\Rightarrow \lim_{x\to 0} f(x) = \frac{1}{2} \times \frac{\left(\lim_{x\to 0}\frac{e^x-1}{x}\right)}{\left(\lim_{x\to 0}\frac{\log(1+2x)}{2x}\right)} = \frac{1}{2} \times \frac{1}{1} = \frac{1}{2}$$

And we have f (0) = 7

$$\Rightarrow \lim_{x\to 0} f(x) \neq f(0)$$

Hence f (x) is discontinuous at x = 0

(v) $f(x) = \begin{cases} \frac{1-x^n}{1-x}, & x \neq 1 \\ n-1, & x = 1 \end{cases} n \in N \; at x = 1$

Solution:

Given

$$f(x) = \begin{cases} \frac{1-x^n}{1-x}, & x \neq 1 \\ n-1, & x = 1 \end{cases} n \in N$$

Clearly, f (1) = n – 1

LHL = $\lim_{h\to 0} f(1-h) = \lim_{h\to 0} \frac{1-(1-h)^n}{1-(1-h)} = \lim_{h\to 0} \frac{1-(1-h)^n}{h}$

Using binomial theorem we get

$$(1-h)^n = \sum_{k=0}^{n} \binom{n}{k} (-h)^k 1^{n-k}$$

$(1-h)^n = 1 - n\,h + \binom{n}{2} h^2 - \ldots\ldots$

LHL =

$\lim_{h\to 0} \frac{1-1+nh-\binom{n}{2}h^2+\cdots \text{higher deg terms}}{h} = \lim_{h\to 0}\{n - \binom{n}{2}h + \binom{n}{3}h^2 - \cdots \text{higher deg terms}\}$

Putting h=0 we get,

LHL = n

RHL = $\lim_{h\to 0} f(1+h) = \lim_{h\to 0}\frac{1-(1+h)^n}{1-(1+h)} = \lim_{h\to 0}\frac{1-(1+h)^n}{-h}$

Using binomial expansion as used above we get the following expression

Similarly,

RHL =

$\lim_{h\to 0}\frac{1-1-nh-\binom{n}{2}h^2-\cdots\text{higher deg terms}}{-h} = \lim_{h\to 0}\{n+\binom{n}{2}h+\binom{n}{3}h^2-\cdots\text{higher deg terms}\}$

Putting h=0 we get,

RHL = n

Thus RHL = LHL ≠ f (1)

Hence f (x) is discontinuous at x=1

(vi) $f(x) = \begin{cases}\frac{|x^2-1|}{x-1}, for & x\neq 1\\ 2, for & x=1\end{cases}$ $at x = 1$

Solution:

Given

$f(x) = \begin{cases}\frac{|x^2-1|}{x-1}, for & x\neq 1\\ 2, for & x=1\end{cases}$

Clearly, f (1) = 2

LHL = $\lim_{h\to 0} f(1-h) = \lim_{h\to 0}\frac{|(1-h)^2-1|}{1-h-1} = \lim_{h\to 0}\frac{|1+h^2-2h-1|}{-h} = \lim_{h\to 0}\frac{|h(h-2)|}{-h}$

Since h is positive no which is very close to 0

∴ (h−2) is negative and hence h (h−2) is also negative.

|h (h−2)| = −h (h−2)

LHL = $\lim_{h\to 0} \frac{-h(h-2)}{-h} = \lim_{h\to 0}(h-2) = -2$

RHL = $\lim_{h\to 0} f(1+h) = \lim_{h\to 0} \frac{|(1+h)^2-1|}{1+h-1} = \lim_{h\to 0} \frac{|1+h^2+2h-1|}{h} = \lim_{h\to 0} \frac{|h(h+2)|}{h}$

Since h is a positive no which is very close to 0

(h+2) is positive and hence h (h−2) is also positive.

∴ |h (h+2)| = h (h+2)

∴ RHL = $\lim_{h\to 0} \frac{h(h+2)}{h} = \lim_{h\to 0}(h+2) = 2$

Clearly, LHL ≠ RHL

Hence f(x) is discontinuous at x=1

(vii) $f(x) = \begin{cases} \frac{2|x|+x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

Solution:

Given

$f(x) = \begin{cases} \frac{2|x|+x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Clearly, f (0) = 0

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{2|-h|+(-h)^2}{-h}$

= $\lim_{h\to 0} \frac{2h+h^2}{-h} = \lim_{h\to 0}(-2-h) = -2$

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{2|h|+(h)^2}{h}$

= $\lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2$

Clearly, LHL ≠ RHL ≠ f (0)

Hence f(x) is discontinuous at x=0

$$(viii)\ f(x) = \begin{cases} |x-a| sin \frac{1}{x-a},\ when\ x \neq a \\ 0,\ when x = a. \end{cases}\ at\ x = a$$

Solution:
Given

$$f(x) = \begin{cases} |x-a| sin \frac{1}{x-a},\ when\ x \neq a \\ 0,\ when x = a. \end{cases}$$

Clearly, f (a) = 0

LHL = $\lim_{h\to 0} f(a-h) = \lim_{h\to 0} |(a-h-a)| \sin\left(\frac{1}{a-h-a}\right)$

= $\lim_{h\to 0} |-h| \sin\left(\frac{1}{-h}\right) = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$

RHL = $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} |a+h-a| \sin\left(\frac{1}{a+h-a}\right) = \lim_{h\to 0} |h| \sin\left(\frac{1}{h}\right)$

= $\lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$

Since whatever is value of h, sin (1/h) is going to range from −1 to 1

As h→ 0, i.e. approximately 0

Clearly, LHL = RHL = f (a)

Hence f (x) is continuous at x = 0

$$11.\ Show\ that\ f(x) = \begin{cases} 1 + x^2,\ if\ 0 \le x \le 1 \\ 2 - x,\ if\ x > 1. \end{cases}\ is\ discontinuous\ at\ x = 1$$

Solution:
Given

$$f(x) = \begin{cases} 1 + x^2, if & 0 \le x \le 1 \\ 2 - x, if & x > 1 \end{cases}$$

Consider LHL at x = 1

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1 - h)$$

$$= \lim_{h \to 0} \left(1 + (1 - h)^2\right) = \lim_{h \to 0} (2 + h^2 - 2h) = 2$$

Now again consider RHL at x = 1

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1 + h)$$

$$= \lim_{h \to 0} (2 - (1 + h)) = \lim_{h \to 0} (1 - h) = 1$$

$$\lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$$

Hence f (x) is discontinuous at x = 1

12. $Show\ that\ f(x) = \begin{cases} \frac{sin3x}{tan2x},\ if\ x < 0 \\ \frac{3}{2},\ if\ x = 0 \\ \frac{\log(1+3x)}{e^{2x} - 1, \ if\ x > 0} \end{cases} \quad is\ continuous\ at\ x = 0$

Solution:

Given

$$f(x) = \begin{cases} \frac{sin3x}{tan2x},\ if\ x < 0 \\ \frac{3}{2},\ if\ x = 0 \\ \frac{\log(1+3x)}{e^{2x} - 1, \ if\ x > 0} \end{cases}$$

Consider LHL at x = 0

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0}\left(\frac{\sin 3(-h)}{\tan 2(-h)}\right) = \lim_{h\to 0}\left(\frac{\sin 3h}{\tan 2h}\right) = \lim_{h\to 0}\left(\frac{\frac{3\sin 3h}{3h}}{\frac{2\tan 2h}{2h}}\right)$$

$$= \frac{\lim_{h\to 0}\left(\frac{3\sin 3h}{3h}\right)}{\lim_{h\to 0}\left(\frac{2\tan 2h}{2h}\right)} = \frac{3\lim_{h\to 0}\left(\frac{\sin 3h}{3h}\right)}{2\lim_{h\to 0}\left(\frac{\tan 2h}{2h}\right)} = \frac{3\times 1}{2\times 1} = \frac{3}{2}$$

Consider RHL at x = 0

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0}\left(\frac{\log(1+3h)}{e^{2h}-1}\right) = \lim_{h\to 0}\left(\frac{3h\frac{\log(1+3h)}{3h}}{\frac{2h(e^{2h}-1)}{2h}}\right)$$

$$= \frac{3}{2}\lim_{h\to 0}\left(\frac{\frac{\log(1+3h)}{3h}}{\frac{(e^{2h}-1)}{2h}}\right) = \frac{3}{2}\,\frac{\lim_{h\to 0}\left(\frac{\log(1+3h)}{3h}\right)}{\lim_{h\to 0}\left(\frac{(e^{2h}-1)}{2h}\right)} = \frac{3\times 1}{2\times 1} = \frac{3}{2}$$

We have f(0) = 3/2

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

Thus f (x) is continuous at x = 0

13. Find the value of a for which the function f defined by

$$f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), x \le 0 \\ \frac{\tan x - \sin x}{x^3}, x > 0 \end{cases} \quad is\ continuous\ at\ x = 0$$

Solution:

Given

$$f(x)=\begin{cases} a\sin\frac{\pi}{2}(x+1), x\le 0 \\ \frac{\tan x-\sin x}{x^3}, x>0 \end{cases}$$

Consider LHL at x = 0

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)=\lim_{h\to 0} a\sin\frac{\pi}{2}(-h+1)=a\sin\frac{\pi}{2}=a$$

Now again consider RHL at x = 0

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0}\frac{\tan h-\sin h}{h^3}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{\frac{\sin h}{\cos h}-\sin h}{h^3}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{\frac{\sin h}{\cos h}(1-\cos h)}{h^3}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{(1-\cos h)\tan h}{h^3}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}\tan h}{4\frac{h^2}{4}\times h}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\frac{2}{4}\lim_{h\to 0}\frac{\sin^2\frac{h}{2}\tan h}{\frac{h^2}{4}\times h}$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\frac{1}{2}\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\lim_{h\to 0}\frac{\tan h}{h}$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \frac{1}{2} \times 1 \times 1$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \frac{1}{2}$$

If f (x) is continuous at x = 0, then

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

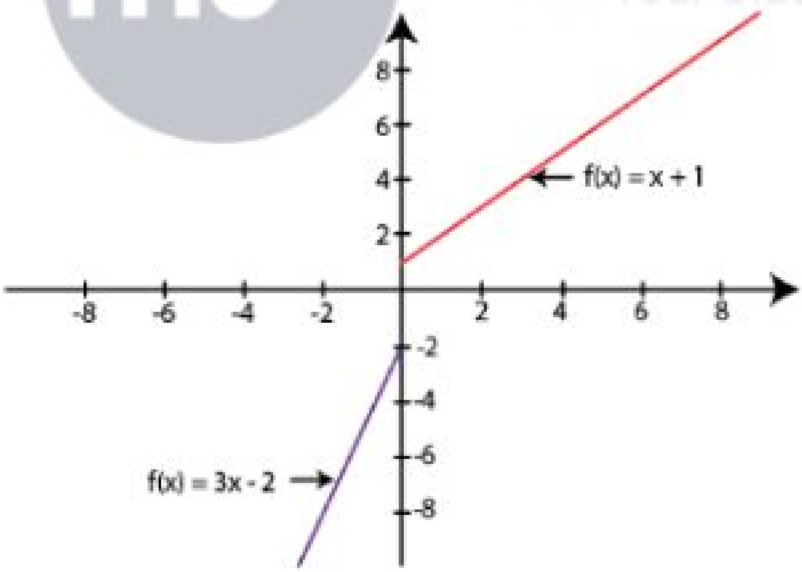
$$\Rightarrow a = \frac{1}{2}$$

14. Examine the continuity of the function

$$f(x) = \begin{cases} 3x - 2, x \leq 0 \\ x + 1, x > 0 \end{cases} \quad at\ x = 0$$

Also sketch the graph of this function.

Solution:



Given

$$f(x) = \begin{cases} 3x - 2, x \le 0 \\ x + 1, x > 0 \end{cases} \quad at\ x = 0$$

The given function can be written as

$$f(x) = \begin{cases} 3x - 2, x < 0 \\ 3(0) - 2, x = 0 \\ x + 1, x > 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 3x - 2, x < 0 \\ -2, x = 0 \\ x + 1, x > 0 \end{cases}$$

Consider LHL at x = 0

$$= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$\lim_{h \to 0} 3(-h) - 2 = -2$$

Now again consider RHL at x = 0

$$= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$\lim_{h \to 0} (h + 1) = 1$$

$$\therefore \lim_{x \to 0^-} f(x) \ne \lim_{x \to 0^+} f(x)$$

Hence f (x) is discontinuous at x = 0

15. Discuss the continuity of the function

$$f(x) = \begin{cases} x, x > 0 \\ 1, x = 0 \\ -x, x < 0 \end{cases} \quad at\ the\ point\ x = 0$$

Solution:

Given

$$f(x)=\begin{cases} x, x>0 \\ 1, x=0 \\ -x, x<0 \end{cases} \quad at\ the\ point\ x=0$$

Consider LHL at x = 0

$$= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$\lim_{h\to 0} -(-h) = 0$$

Consider RHL at x = 0

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$\lim_{h\to 0} (h) = 0$$

And we have f (0) = 1

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \neq f(0)$$

Hence f (x) is discontinuous at x = 0

16. Discuss the continuity of the function

$$f(x)=\begin{cases} x, 0\le x<\frac{1}{2} \\ 12, x=\frac{1}{2} \\ 1-x, \frac{1}{2}<x\le 1 \end{cases} \quad at\ the\ point\ x=\frac{1}{2}$$

Solution:

Given

$$f(x)=\begin{cases} x, 0\le x<\frac{1}{2} \\ 12, x=\frac{1}{2} \\ 1-x, \frac{1}{2}<x\le 1 \end{cases}$$

Consider LHL at x = ½

$$\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$$

$$\lim_{h \to 0} \left(\frac{1}{2} - h\right) = \frac{1}{2}$$

Again consider RHL at x = ½

$$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$$

$$\lim_{h \to 0} \left(1 - \left(\frac{1}{2} + h\right)\right) = \frac{1}{2}$$

We have f (1/2) = ½

$$\therefore \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^+} f(x) = f\left(\frac{1}{2}\right)$$

Hence f (x) is continuous at x = ½

17. Discuss the continuity of

$$f(x) = \begin{cases} 2x - 1, x < 0 \\ 2x + 1, x \geq 0 \end{cases} \; at x = 0$$

Solution:

Given

$$f(x) = \begin{cases} 2x - 1, x < 0 \\ 2x + 1, x \geq 0 \end{cases} \; at x = 0$$

Consider LHL at x = 0

$$\lim_{x \to 0^-} f(x) = 2(0) - 1 = -1$$

Again consider RHL at x = 0

$$\lim_{x \to 0^+} f(x) = 2(0) + 1 = 1$$

$$\Rightarrow \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

Hence f (x) is discontinuous at x = 0

18. For what value of k is the function

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, x \neq 1 \\ k, x = 1 \end{cases} \ continuous\ at\ x = 1?$$

Solution:
Given

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, x \neq 1 \\ k, x = 1 \end{cases}$$

If f (x) is continuous at x = 1, then

$$\lim_{x \to 1} f(x) = f(1)$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = k$$

$$\lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = k$$

$$\lim_{x \to 1} (x + 1) = k$$

$$k = 2$$

19. Determine the value of the constant k so that the function

$$f(x)=\begin{cases}\frac{x^2-3x+2}{x-1}, x\neq 1\\ k, x=1\end{cases}\ continuous\ at\ x=1$$

Solution:

Given

$$f(x)=\begin{cases}\frac{x^2-3x+2}{x-1}, x\neq 1\\ k, x=1\end{cases}$$

If f (x) is continuous at x = 1, then

$$\lim_{x\to 1} f(x)=f(1)$$

$$\lim_{x\to 1}\frac{x^2-3x+2}{x-1}=k$$

$$\lim_{x\to 1}\frac{(x-2)(x-1)}{x-1}=k$$

$$\lim_{x\to 1}(x-2)=k$$

$$k=-1$$

20. For what value of k is the function

$$f(x)=\begin{cases}\frac{sin\ 5x}{3x},\ if\ x\neq 0\\ k,\ if\ x=0\end{cases}\ continuous\ at\ x=0?$$

Solution:

Given

$$f(x)=\begin{cases}\frac{sin\ 5x}{3x},\ if\ x\neq 0\\ k,\ if\ x=0\end{cases}$$

If f (x) is continuous at x = 0, then we have

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = k$$

$$\lim_{x\to 0} \frac{5\sin 5x}{3\times 5x} = k$$

$$\frac{5}{3}\lim_{x\to 0} \frac{\sin 5x}{5x} = k$$

$$\frac{5}{3}\times 1 = k$$

$$k = \frac{5}{3}$$

21. Determine the value of the constant k so that the function
$$f(x) = \begin{cases} kx^2, & if\ x \le 2 \\ 3, & if\ x > 2 \end{cases}\quad is\ continuous\ at\ x = 2$$

Solution:

Given

$$f(x) = \begin{cases} kx^2, & if\ x \le 2 \\ 3, & if\ x > 2 \end{cases}$$

If f (x) is continuous at x = 2, then we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

Now,

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} k(2-h)^2 = 4k$$

$f(2) = 3$

From the above equation we can write as

$4k = 3$

$\Rightarrow k = \frac{3}{4}$

22. Determine the value of the constant k so that the function

$$f(x) = \begin{cases} \frac{sin\ 2x}{5x},\ if\ x \neq 0 \\ k,\ if\ x = 0 \end{cases}\ is\ continuous\ at\ x = 0$$

Solution:

Given

$$f(x) = \begin{cases} \frac{sin\ 2x}{5x},\ if\ x \neq 0 \\ k,\ if\ x = 0 \end{cases}$$

If f (x) is continuous at x = 0

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0} \frac{\sin 2x}{5x} = k$$

$$\lim_{x \to 0} \frac{2 \sin 2x}{5 \times 2x} = k$$

$$\frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} = k$$

$$\frac{2}{5} \times 1 = k$$

$$k = \frac{2}{5}$$

23. Find the values of a so that the function

$$f(x) = \begin{cases} ax+5, \ if \ x \le 2 \\ x-1, \ if \ x > 2 \end{cases} \ is \ continuous \ at \ x = 2$$

Solution:

Given

$$f(x) = \begin{cases} ax+5, \ if \ x \le 2 \\ x-1, \ if \ x > 2 \end{cases}$$

Consider LHL at x = 2

$$\lim_{h \to 0} a(2-h)+5 = 2a+5$$

Now again consider

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h)$$

$$\lim_{h \to 0} (2+h-1)$$

= 1

$$f(2) = a(2)+5 = 2a+5$$

Since f (x) is continuous at x = 2 we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$2a+5 = 1$$

$$2a = -4$$

$$a = -2$$