$$= \frac{1}{ab} tan^{-1}\left(\frac{b}{a} tanx\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2ab}$$

26. Question

Evaluate the following integrals

$$\int_0^{\pi/2} \frac{dx}{\left(1+\cos^2 x\right)}$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+cos^2x} dx$

Dividing by $\cos^2 x$ in numerator and denominator, we get

$$I = \int_0^{\frac{\pi}{2}} \frac{sec^2x}{sec^2x + tan^2x} dx = \int_0^{\frac{\pi}{2}} \frac{sec^2x}{1 + 2tan^2x} dx$$

Consider $I = \int_0^{\frac{\pi}{2}} \frac{sec^2x}{a^2+b^2tan^2x} dx$

Let tan x=t

$\Rightarrow sec^2 x dx = dt$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{a^2 + b^2t^2} dt$$

$$= \frac{1}{b^2} \int_0^{\frac{\pi}{2}} \frac{1}{\frac{a^2}{b^2} + t^2} dt$$

Let $t = \frac{a}{b} tan\theta$

=tan x

$$I = \frac{1}{b^2} \int_0^{\frac{\pi}{2}} \frac{\frac{a}{b} sec^2\theta}{\frac{a^2}{b^2} + \frac{a^2}{b^2} tan^2\theta} d\theta$$

$$= \frac{1}{ab}\theta = \frac{1}{ab} tan^{-1}\left(\frac{b}{a} tanx\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2ab}$$

Here, a=1 and b=√2

Hence,

$$I = \frac{\pi}{2\sqrt{2}}$$

27. Question

Evaluate the following integrals

$$\int_0^{\pi/2} \frac{dx}{\left(4+9\cos^2 x\right)}$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{4+9cos^2 x} dx$

Dividing by $\cos^2 x$ in numerator and denominator, we get

$$I = \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{4sec^2 x + 9tan^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{4 + 13tan^2 x} dx$$

Consider $I = \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{a^2 + b^2 tan^2 x} dx$

Let tan x=t

$\Rightarrow sec^2 x dx = dt$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{a^2 + b^2 t^2} dt$$

$$= \frac{1}{b^2} \int_0^{\frac{\pi}{2}} \frac{1}{\frac{a^2}{b^2} + t^2} dt$$

Let $t = \frac{a}{b} tan\theta$

=tan x

$$I = \frac{1}{b^2} \int_0^{\frac{\pi}{2}} \frac{\frac{a}{b} sec^2\theta}{\frac{a^2}{b^2} + \frac{a^2}{b^2} tan^2\theta} d\theta$$

$$= \frac{1}{ab} \theta$$

$$= \frac{1}{ab} tan^{-1}\left(\frac{b}{a} tanx\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2ab}$$

Here, a=2 and b=√13

Hence,

$$I = \frac{\pi}{4\sqrt{13}}$$

28. Question

Evaluate the following integrals

$$\int_0^{\pi/2} \frac{dx}{(5+4\sin x)}$$

Answer

Let $I=\int_0^{\frac{\pi}{2}}\frac{1}{5+4sinx}dx$

Using $sinx=\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$, we get

$$I=\int_0^{\frac{\pi}{2}}\frac{1}{5+4\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2\left(\frac{x}{2}\right)}{5+5\tan^2\left(\frac{x}{2}\right)+8\tan\left(\frac{x}{2}\right)}dx$$

Let $tan\left(\frac{x}{2}\right)=t$

$\Rightarrow\frac{1}{2}sec^2\left(\frac{x}{2}\right)dx=dt,$

when x=0, t=0 and when $x=\frac{\pi}{2}$, t=1.

Hence, $I=\int_0^1\frac{2}{5+5t^2+8t}dt$

$$=\frac{2}{5}\int_0^1\frac{1}{t^2+\frac{8}{5}t+\frac{16}{25}+\frac{9}{25}}dt$$

$$=\frac{2}{5}\int_0^1\frac{1}{\left(t+\frac{4}{5}\right)^2+\frac{9}{25}}dt$$

Let $t+\frac{4}{5}=u$

$\Rightarrow$ dt=du.

When t=0, $u=\frac{4}{5}$ and when t=1, $u=\frac{9}{5}$.

$$I=\frac{2}{5}\int_{\frac{4}{5}}^{\frac{9}{5}}\frac{1}{(u)^2+\frac{9}{25}}du$$

$$=\frac{2}{5}\times\frac{5}{3}tan^{-1}\left(\frac{5x}{3}\right)\Bigg|_{\frac{4}{5}}^{\frac{9}{5}}$$

$$=\frac{2}{3}\left(tan^{-1}3-tan^{-1}\left(\frac{4}{3}\right)\right)$$

$$=\frac{2}{3}\times tan^{-1}\left(\frac{3-\frac{4}{3}}{5}\right)$$

$$=\frac{2}{3}tan^{-1}\left(\frac{1}{3}\right)$$

$$\left(Using\ tan^{-1}x-tan^{-1}y=tan^{-1}\left(\frac{x-y}{1+xy}\right)\right)$$

29. Question

Evaluate the following integrals

$$\int_0^{\pi} \frac{dx}{(6-\cos x)}$$

Answer

Let $I = \int_0^{\pi} \frac{1}{6-cosx} dx$

Using $cosx = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$, we get

$$I = \int_0^{\pi} \frac{1}{6 - \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}} dx$$

$$= \int_0^{\pi} \frac{\sec^2\left(\frac{x}{2}\right)}{5+7\tan^2\left(\frac{x}{2}\right)} dx$$

Let $tan\left(\frac{x}{2}\right) = t$

$\Rightarrow \frac{1}{2} sec^2\left(\frac{x}{2}\right) dx = dt,$

when x=0, t=0 and when x=π, t=∞.

Hence, $I = \int_0^{\infty} \frac{2}{5+7t^2} dt$

$$= \frac{2}{7}\int_0^{\infty} \frac{1}{t^2+\frac{5}{7}} dt$$

$$= \frac{2}{7} \times \sqrt{\frac{7}{5}} tan^{-1}\left(\sqrt{\frac{7}{5}}x\right)\Big|_0^{\infty}$$

$$\Rightarrow I = \frac{2}{\sqrt{35}}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{\sqrt{35}}$$

30. Question

Evaluate the following integrals

$$\int_0^{\pi} \frac{dx}{(5+4\cos x)}$$

Answer

Let $I = \int_0^{\pi} \frac{1}{5+4cosx} dx$

Using $cosx = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$, we get

$$I=\int_0^{\pi}\frac{1}{5+4\times\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}}dx$$

$$=\int_0^{\pi}\frac{\sec^2\left(\frac{x}{2}\right)}{9+\tan^2\left(\frac{x}{2}\right)}dx$$

Let $tan\left(\frac{x}{2}\right)=t$

$\Rightarrow\frac{1}{2}sec^2\left(\frac{x}{2}\right)dx=dt,$

when x=0, t=0 and when x=π, t=∞.

Hence, $I=\int_0^{\infty}\frac{2}{9+t^2}dt$

$$=2\int_0^{\infty}\frac{1}{9+t^2}dt$$

$$=2\times\frac{1}{3}tan^{-1}\left(\frac{x}{3}\right)\Big|_0^{\infty}$$

$$\Rightarrow I=\frac{2}{3}\left(\frac{\pi}{2}-0\right)$$

$$=\frac{\pi}{3}$$

31. Question

Evaluate the following integrals

$$\int_0^{\pi/2}\frac{dx}{(\cos x+2\sin x)}$$

Answer

Let $I=\int_0^{\frac{\pi}{2}}\frac{1}{cosx+2sinx}dx$

Using $sinx=\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

And

$cosx=\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)},$

we get

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{1}{\frac{1-tan^2\left(\frac{x}{2}\right)}{1+tan^2\left(\frac{x}{2}\right)}+2\frac{2\,tan\left(\frac{x}{2}\right)}{1+tan^2\left(\frac{x}{2}\right)}}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)+4\tan\left(\frac{x}{2}\right)}dx$$

Let $tan\left(\frac{x}{2}\right)=t$

$\Rightarrow \frac{1}{2}sec^2\left(\frac{x}{2}\right)dx = dt,$

when x=0, t=0

and when $x = \frac{\pi}{2}$, t=1.

Hence,

$I = \int_0^1 \frac{2}{1-t^2+4t}dt$

$= -2\int_0^1 \frac{1}{t^2-4t+4-5}dt$

$= -2\int_0^1 \frac{1}{(t-2)^2-5}dt$

Let t-2=u

⇒ dt=du.

Also, when t=0, u=-2

and when t=1, u=-1.

$\Rightarrow\ I = -2\int_{-2}^{-1} \frac{1}{u^2-5}dt$

$= -2 \times \frac{1}{2\sqrt{5}}\log_e\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| \Big|_{-2}^{-1}$

$\left(Using\ \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}log_e\left|\frac{x-a}{x+a}\right|\right)$

Hence,

$I = -\frac{1}{\sqrt{5}}\left(\log_e\left|\frac{-1-\sqrt{5}}{-1+\sqrt{5}}\right| - \log_e\left|\frac{-2-\sqrt{5}}{-2+\sqrt{5}}\right|\right)$

$= \frac{-1}{\sqrt{5}}\left(\log_e\left|\frac{\sqrt{5}+1}{\sqrt{5}-1}\right| \times \left|\frac{\sqrt{5}-2}{2+\sqrt{5}}\right|\right)$

$\left(Using\ log_e a - log_e b = log_e\frac{a}{b}\right)$

$\Rightarrow\ I = \frac{-1}{\sqrt{5}}\left(log_e\left|\frac{3-\sqrt{5}}{3+\sqrt{5}}\right|\right)$

$= \frac{-2}{\sqrt{5}}\left(\log_e\left(\frac{3-\sqrt{5}}{2}\right)\right)$

(Using $\log_e a^b = b\log_e a$)

32. Question

Evaluate the following integrals

$$\int_0^{\pi} \frac{dx}{(3+2\sin x+\cos x)}$$

Answer

Let $I = \int_0^{\pi} \frac{1}{3+cosx+2sinx}dx$

Using $sinx = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

And

$cosx = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$,

we get

$$\Rightarrow I = \int_0^{\pi} \frac{1}{3 + \frac{1-tan^2\left(\frac{x}{2}\right)}{1+tan^2\left(\frac{x}{2}\right)} + 2\frac{2\,tan\left(\frac{x}{2}\right)}{1+tan^2\left(\frac{x}{2}\right)}} dx$$

$$= \int_0^{\pi} \frac{\sec^2\left(\frac{x}{2}\right)}{4 + 2\tan^2\left(\frac{x}{2}\right) + 4\tan\left(\frac{x}{2}\right)} dx$$

Let $tan\left(\frac{x}{2}\right) = t$

$$\Rightarrow \frac{1}{2}sec^2\left(\frac{x}{2}\right)dx = dt,$$

when x=0, t=0

and when $x = \pi$, t=∞.

Hence,

$I = \int_0^{\infty} \frac{1}{(t+1)^2+1} dt$

Let t+1=u

⇒ dt=du.

Also, when t=0, u=1

and when t=∞, u=∞.

$$I = \int_1^{\infty} \frac{1}{u^2+1} dt$$

$$= tan^{-1}u\Big|_1^{\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

33. Question

Evaluate the following integrals

$$\int_0^{\pi/4} \frac{\tan^3 x}{(1+\cos 2x)} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \frac{tan^3 x}{1+cos2x} dx$

Using $1+\cos 2x = 2\cos^2 x$, we get

$I=\frac{1}{2}\int_0^{\frac{\pi}{4}} tan^3 x sec^2 x\, dx$

Let tan x=t

⇒ $sec^2 xdx=dt$.

when x=0, t=0

and when $x=\frac{\pi}{4}$, t=1.

$=\frac{1}{2}\int_0^1 t^3\,dt=\frac{t^4}{8}\Big|_0^1$

$=\frac{1}{8}$

34. Question

Evaluate the following integrals

$$\int_0^{\pi/2} \frac{\sin x \cos x}{\left(\cos^2 x+3\cos x+2\right)}dx$$

Answer

Let $I=\int_0^{\frac{\pi}{2}} \frac{sinxcosx}{cos^2x+3cosx+2}dx$

Let cos x=t

⇒ -sin x dx=dt.

Also, when x=0, t=1

and when $x=\frac{\pi}{2}$, t=0.

Hence,

$I=-\int_1^0 \frac{t}{t^2+3t+2}dt$

$=-\int_1^0 \frac{2(t+1)-(t+2)}{(t+1)(t+2)}dt$

$=-\int_1^0 \frac{2}{(t+2)}dt+\int_1^0 \frac{1}{(t+1)}dt$

$\Rightarrow\ I=-2\,log_e(t+2)\Big|_1^0+log_e(t+1)\Big|_1^0$

$=-2\log_e 2+2\log_e 3-\log_e 2$

Hence $I=\log_e 9-\log_e 8$

$(Using\ b\,log_e a=log_e a^b\ and\ log_e a+log_e b=log_e ab)$

35. Question

Evaluate the following integrals

$$\int_0^{\pi/2} \frac{\sin 2x}{\left(\sin^4 x+\cos^4 x\right)}dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{sin2x}{sin^4x+cos^4x}dx$

Using sin 2x =2 sin x cos x, we get

$$I = \int_0^{\frac{\pi}{2}} \frac{2sinxcosx}{cos^4x(tan^4x+1)}dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{tanxsec^2x}{(tan^4x+1)}dx$$

Let tan x=t

⇒ $sec^2xdx=dt$.

Also, when x=0, t=0

and when $x = \frac{\pi}{2}$, t=∞.

Hence, $2\int_0^{\infty} \frac{t}{(t^4+1)}dt$

Let $x^2=t$

⇒ 2xdx=dt.

Also, when x=0, t=0

and when x=∞, t=∞.

Hence, $I = \int_0^{\infty} \frac{1}{1+t^2}dt$

$$= tan^{-1}t\Big|_0^{\infty}$$

$$= \frac{\pi}{2}$$

36. Question

Evaluate the following integrals

$$\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}dx$$

Answer

Let $I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+cosx}}{(1-cosx)^{\frac{5}{2}}}dx$

Using $1 + cosx = 2cos^2\left(\frac{x}{2}\right)$

And

$1 - cosx = 2sin^2\left(\frac{x}{2}\right)$,

we get

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{2}\cos\left(\frac{x}{2}\right)}{4\sqrt{2}\left(\sin\left(\frac{x}{2}\right)\right)^5}dx$$

$= \frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\left(\frac{x}{2}\right) cosec^4\left(\frac{x}{2}\right) dx$

Let $\cot\left(\frac{x}{2}\right) = t$

$\Rightarrow -\frac{1}{2} cosec^2\left(\frac{x}{2}\right) dx = dt.$

Also, when $x = \frac{\pi}{3}, t = \sqrt{3}$

and when $x = \frac{\pi}{2}$, t=1

Hence,

$I = -\frac{1}{2}\int_{\sqrt{3}}^{1} t\,(1 + t^2)dt$

$= -\frac{1}{2}\frac{t^2}{2}\Big|_{\sqrt{3}}^{1} - \frac{1}{2}\frac{t^4}{4}\Big|_{\sqrt{3}}^{1}$

$= \frac{1}{2} + 1$

$= \frac{3}{2}$

37. Question

Evaluate the following integrals

$\int_{0}^{1}\left(\cos^{-1} x\right)^2 dx$

Answer

Let $I = \int_0^1 (cos^{-1}x)^2\, dx$

Let x=cost ⇒ dx=-sin t dt.

Also, when x=0, $t = \frac{\pi}{2}$

and when x=1, t=0.

Hence, $I = -\int_{\frac{\pi}{2}}^{0} t^2 sint\, dt$

Using integration by parts, we get

$I = -\left(t^2 \times -cost\,\Big|_{\frac{\pi}{2}}^{0} + 2\int_{\frac{\pi}{2}}^{0} tcost\, dt\right)$

$= -\left(0 - 0 + 2t \times sint\,\Big|_{\frac{\pi}{2}}^{0} - 2\int_{\frac{\pi}{2}}^{0} sint\, dt\right)$

$= -\left(-\pi + 2cost\,\Big|_{\frac{\pi}{2}}^{0}\right)$

Hence, I=π-2

38. Question

Evaluate the following integrals

$$\int_0^1 x\left(\tan^{-1}x\right)^2 dx$$

Answer

Let $I = \int_0^1 x(tan^{-1}x)^2\,dx$

Using integration by parts, we get

$$I = \frac{(tan^{-1}x)^2x^2}{2}\Big|_0^1 - \int_0^1 \frac{2tan^{-1}x}{1+x^2}\times\frac{x^2}{2}dx$$

$$= \frac{\pi^2}{32} - 0 - \int_0^1 \frac{tan^{-1}x}{1+x^2}\times(1+x^2-1)\,dx$$

$$= \frac{\pi^2}{32} - \int_0^1 tan^{-1}x dx + \int_0^1 \frac{tan^{-1}x}{1+x^2}dx$$

Let $\tan^{-1}x=t$

$\Rightarrow \frac{1}{1+x^2}dx = dt.$

When x=0, t=0 and when x=1, $t = \frac{\pi}{4}$.

Hence

$$I = \frac{\pi^2}{32} - tan^{-1}x\times x\Big|_0^1 + \int_0^1 \frac{x}{1+x^2}dx + \int_0^{\frac{\pi}{4}} tdt$$

$$= \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{t^2}{2}\Big|_0^{\frac{\pi}{4}} + \int_0^1 \frac{x}{1+x^2}dx$$

Let $1+x^2=y$

$\Rightarrow$ 2xdx=dy.

Also, when x=0, y=1

and when x=1, y=2.

$$I = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\int_1^2 \frac{1}{y}dy$$

$$= \frac{\pi}{4}\left(\frac{\pi}{4} - 1\right) + \frac{1}{2}\log_e y\Big|_1^2$$

$$= \frac{\pi}{4}\left(\frac{\pi}{4} - 1\right) + \frac{1}{2}\log_e 2.$$

39. Question

Evaluate the following integrals

$$\int_0^1 \sin^{-1}\sqrt{x}\,dx$$

Answer

Let $I = \int_0^1 sin^{-1}\sqrt{x}\,dx$

Let $\sqrt{x}=t$

$\Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

or

dx=2tdt.

When, x=0, t=0

and when x=1, t=1.

Hence,

$I = 2\int_0^1 t\sin^{-1}t\,dt$

Using integration by parts, we get

$I = 2\left(\sin^{-1}t \times \frac{t^2}{2}\Big|_0^1 - \int_0^1 \frac{1}{\sqrt{1-t^2}} \times \frac{t^2}{2}dt\right)$

$= \frac{\pi}{2} - \int_0^1 \frac{t^2}{\sqrt{1-t^2}}dt$

Let t=sin y

⇒ dt=cos y dy.

When t=0, y=0, when t=1, $y = \frac{\pi}{2}$.

$I = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin^2 y dy$ ..... (1)

Using, $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we get

$I = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \cos^2 y dy$ .....(2)

Adding (1) and (2), we get

$2I = \pi - \int_0^{\frac{\pi}{2}} dy$

$= \pi - \frac{\pi}{2}$

Hence,

$I = \frac{\pi}{4}$

40. Question

Evaluate the following integrals

$$\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}dx$$

Answer

Let $I = \int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}dx$

Let x=a $\tan^2$y

⇒ dx=2a tan y $\sec^2$y dy.

Also, when x=0, y=0

and when x=a, $y=\frac{\pi}{4}$

Hence $I=\int_0^{\frac{\pi}{4}} sin^{-1}\left(\sqrt{\frac{atan^2 y}{a+atan^2 y}}\right) 2a\tan y\, sec^2\, y\, dy = 2a\int_0^{\frac{\pi}{4}} y\tan y\; sec^2 y dy$

Using integration by parts, we get

$$I=2a\left(y\int_0^{\frac{\pi}{4}} tany sec^2 y dy-\int_0^{\frac{\pi}{4}}\left(\int tany sec^2 y dy\right)dy\right)$$

Let tan y=t

⇒ $sec^2 ydy=dt$.

Also, when y=0, t=0

and when $y=\frac{\pi}{4}$, t=1.

Also, $y=tan^{-1}t$

$$\Rightarrow\ dy=\frac{dt}{1+t^2}$$

$$I=2a\left(tan^{-1}t\int tdt\Big|_0^1-\int_0^1\left(\int tdt\right)\frac{dt}{1+t^2}\right)$$

$$=2a\left(\frac{tan^{-1}t\times t^2}{2}\Big|_0^1\right)-2a\int_0^1\frac{t^2}{2}\frac{dt}{1+t^2}$$

$$=\frac{a\pi}{4}-a\int_0^1\frac{t^2}{1+t^2}dt$$

Let $I'=\int_0^1\frac{t^2}{1+t^2}dt$

$$=\int_0^1\frac{1+t^2-1}{1+t^2}dt$$

$$=\int_0^1 dt-\int_0^1\frac{1}{1+t^2}dt$$

$$=t\Big|_0^1-tan^{-1}t\Big|_0^1$$

Hence $I'=1-\frac{\pi}{4}$

Substituting value of I' in I, we get

$$I=\frac{a\pi}{4}-a\left(1-\frac{\pi}{4}\right)$$

$$=a\left(\frac{\pi}{2}-1\right)$$

41. Question

Evaluate the following integrals

$$\int_0^9\frac{dx}{\left(1+\sqrt{x}\right)}$$

Answer

Let $I = \int_0^9 \frac{1}{1+\sqrt{x}} dx$

Let √x=u

$\Rightarrow \frac{1}{2\sqrt{x}} dx = du$

$= \frac{1}{2u} dx$ or dx=2udu.

Also, when x=0, u=0 and x=9, u=3.

Hence,

$I = \int_0^3 \frac{2u}{1+u} du$

$= 2\left(\int_0^3 \frac{u+1-1}{1+u} du\right)$

$= 2\left(\int_0^3 du - \int_0^3 \frac{1}{1+u} du\right)$

$I = 2u \Big|_0^3 - \log_e(1+u) \Big|_0^3$

$= 6 - 2\log_e 4$

$= 6 - 4\log_e 2$

$(Using\ log_e a^b = b\, log_e a)$

42. Question

Evaluate the following integrals

$$\int_0^1 x^3 \sqrt{1+3x^4}\, dx$$

Answer

Let $I = \int_0^1 x^3 \sqrt{1+3x^4} dx$

Let $1+3x^4=t$

$\Rightarrow 12x^3dx=dt$.

Also, when x=0, t=1 and when x=1, t=4.

$I = \frac{1}{12}\int_1^4 \sqrt{t}\, dt$

$= \frac{1}{12} \times \frac{2}{3} t^{\frac{3}{2}} \Big|_1^4$

$= \frac{7}{18}$

43. Question

Evaluate the following integrals

$$\int_0^1 \frac{\left(1-x^2\right)}{\left(1+x^2\right)^2} dx$$

Answer

Let $I = \int_0^1 \frac{1-x^2}{(1+x^2)^2} dx$

Let $I' = \int_0^1 \frac{1}{(1+x^2)^2} dx$

Let x=tan t

⇒ $dx = sec^2 t dt$.

Also when x=0, t=0 and when x=1, $t = \frac{\pi}{4}$.

Hence, $I' = \int_0^{\frac{\pi}{4}} \frac{sec^2 t}{(1+tan^2 t)^2} dt$

$$= \int_0^{\frac{\pi}{4}} cos^2 t dt$$

Using $cos^2 t = \frac{1+cos2t}{2}$, we get

$$I' = \int_0^{\frac{\pi}{4}} \left(\frac{1+cos2t}{2}\right) dt$$

$$= \frac{t}{2}\Big|_0^{\frac{\pi}{4}} + \frac{sin2t}{4}\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi+2}{8}$$

Let $I'' = \int_0^1 \frac{x^2}{(1+x^2)^2} dx$

$$= \int_0^1 x \times \frac{x}{(1+x^2)^2} dx$$

$$= x \int_0^1 \frac{x}{(1+x^2)^2} dx - \int_0^1 \left(\int \frac{x}{(1+x^2)^2} dx\right) dx$$

Let $1+x^2=t$ ⇒ 2xdx=dt.

When x=0, t=1 and when x=1, t=2.

$$I'' = \sqrt{t-1} \times \frac{1}{2}\int_1^2 \frac{1}{t^2} dt - \int_1^2 \frac{\left(\frac{1}{2}\int \frac{1}{t^2} dt\right) dt}{2\sqrt{t-1}}$$

$$= -\frac{\sqrt{t-1}}{2} \times \frac{1}{t}\Big|_1^2 + \int_1^2 \frac{dt}{4t\sqrt{t-1}}$$

$$= -\frac{1}{4} + \int_1^2 \frac{dt}{4t\sqrt{t-1}}$$

Substituting $t=1+x^2$

⇒ 2xdx=dt.

When t=1, x=0 and when t=2, x=1.

$$I'' = -\frac{1}{4} + \int_0^1 \frac{2x dx}{4x(1+x^2)}$$

$$= -\frac{1}{4} + \frac{1}{2} tan^{-1} x \Big|_0^1$$

$$= \frac{\pi - 2}{8}$$

Hence,

$$I = \frac{\pi + 2}{8} - \frac{\pi - 2}{8}$$

$$= \frac{1}{2}$$

44. Question

Evaluate the following integrals

$$\int_1^2 \frac{dx}{(x+1)\sqrt{x^2 - 1}}$$

Answer

Let $I = \int_1^2 \frac{1}{(x+1)\sqrt{x^2-1}} dx$

Let x=sect

⇒ dx=sec t tan t dt.

Also,

when x=1, t=0 and when x=2, $t = \frac{\pi}{3}$

Hence,

$$I = \int_0^{\frac{\pi}{3}} \frac{secttant}{(sect+1)\sqrt{sec^2 t - 1}} dt$$

$$= \int_0^{\frac{\pi}{3}} \frac{sect}{(sect+1)} dt$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{(1+cost)} dt$$

Using $1 + cost = 2cos^2\left(\frac{t}{2}\right)$, we get

$$I = \frac{1}{2}\int_0^{\frac{\pi}{3}} sec^2\left(\frac{t}{2}\right) dt$$

$$= \tan\left(\frac{t}{2}\right)\Big|_0^{\frac{\pi}{3}}$$

$$= \frac{1}{\sqrt{3}}$$

45. Question

Evaluate the following integrals

$$\int_0^{\pi/2}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$$

Answer

Let $I=\int_0^{\frac{\pi}{2}}(\sqrt{tanx}+\sqrt{cotx})dx=\int_0^{\frac{\pi}{2}}\frac{sinx+cosx}{\sqrt{sinxcosx}}dx$

Let sin x- cos x=t

⇒ (cos x + sin x)dx=dt.

When x=0, t=-1 and $x=\frac{\pi}{2}$, t=1.

Also, $t^2=(\sin x - \cos x)^2$

$=\sin^2x+\cos^2x-2\sin x\cos x$

=1-2sinxcosx

or

$$sincosx=\frac{1-t^2}{2}$$

Hence $I=\sqrt{2}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}dt$

Let t=sin y

⇒ dt=cos y dy.

Also, when t=-1, $y=-\frac{\pi}{2}$

and when t=1, $y=\frac{\pi}{2}$.

$$I=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{cosy}{\sqrt{1-sin^2y}}dy$$

$$=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}dy=\pi\sqrt{2}$$

46. Question

Evaluate the following integrals

$$\int_2^3\frac{(2-x)}{\sqrt{5x-6-x^2}}dx$$

Answer

Let $I=\int_2^3\frac{2-x}{\sqrt{5x-6-x^2}}dx$

Let,

$$2-x=a\frac{d}{dx}(5x-6-x^2)+b$$

=-2ax+5a+b

Hence -2a=-1 and 5a+b=2.

Solving these equations,

we get $a = \frac{1}{2}$ and $b = -\frac{1}{2}$.

We get,

$I = \frac{1}{2}\int_2^3 \frac{-2x+5}{\sqrt{5x-6-x^2}} dx - \frac{1}{2}\int_2^3 \frac{1}{\sqrt{5x-6-x^2}} dx$

Let $I' = \int_2^3 \frac{-2x+5}{\sqrt{5x-6-x^2}} dx$

Let $5x-6-x^2=t$

$\Rightarrow (5-2x)\, dx=dt$.

When x=2, t=0 and when x=3, y=0.

Hence $I' = \int_0^0 \frac{1}{\sqrt{t}} dt = 0$

$\left(Since \int_a^a f(x)dx = 0\right)$

Let,

$I'' = \int_2^3 \frac{1}{\sqrt{5x-6-x^2}} dx$

$= \int_2^3 \frac{1}{\sqrt{\frac{1}{4} - \left(x - \frac{5}{2}\right)^2}}$

$= sin^{-1}\left(\frac{x - \frac{5}{2}}{\frac{1}{2}}\right)$

$= sin^{-1}(2x-5)\Big|_2^3$

$=\pi$

Hence,

$I = \frac{1}{2} \times 0 - \frac{1}{2} \times \pi$

$= -\frac{\pi}{2}$

47. Question

Evaluate the following integrals

$$\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^3} d\theta$$

Answer

Let $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{cosx}{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)^3} dx$

Using $cosx = cos^2\left(\frac{x}{2}\right) - sin^2\left(\frac{x}{2}\right)$, we get

$$I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)^{2}}dx$$

Let $\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)=t$

$\Rightarrow \frac{1}{2}\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)dx=dt.$

Also, when $x=\frac{\pi}{4}, t=\cos\left(\frac{\pi}{8}\right)+\sin\left(\frac{\pi}{8}\right)=\alpha(Let)$

and when $x=\frac{\pi}{2}, t=\sqrt{2}$

$$I=\int_{\alpha}^{\sqrt{2}}\frac{2}{t^{2}}dt$$

$$=-2\times\frac{1}{t}\Big|_{\alpha}^{\sqrt{2}}$$

$$=\frac{2}{\cos\left(\frac{\pi}{8}\right)+\sin\left(\frac{\pi}{8}\right)}-\sqrt{2}$$

48. Question

Evaluate the following integrals

$$\int_{0}^{(\pi/2)^{1/3}} x^{2}\sin x^{3}dx$$

Answer

Let $I=\int_{0}^{\left(\frac{\pi}{2}\right)^{\frac{1}{3}}}x^{2}\sin(x^{3})\,dx$

Let x^3=t

$\Rightarrow$ $3x^2$=dt.

Also, when x=0, t=0 and when $x=\left(\frac{\pi}{2}\right)^{\frac{1}{3}}, t=\frac{\pi}{2}.$

Hence, $I=\frac{1}{3}\int_{0}^{\frac{\pi}{2}}\sin(t)\,dt$

$$=\frac{-1}{3}cost\Big|_{0}^{\frac{\pi}{2}}$$

$$=-\frac{1}{3}(0-1)$$

$$=\frac{1}{3}$$

49. Question

Evaluate the following integrals

$$\int_{1}^{2}\frac{dx}{x(1+\log x)^{2}}$$

Answer

Let $I=\int_1^2 \frac{1}{x(1+\log_e x)^2}dx$

Let $1+\log_e x=t$

$\Rightarrow \frac{1}{x}dx=dt.$

Also, when x=1, t=1 and when x=2, $t=1+\log_e 2$

Hence $I=\int_1^{1+\log_e 2}\frac{1}{t^2}dt$

$=-\frac{1}{t}\Big|_1^{1+\log_e 2}$

$=1-\frac{1}{1+\log_e 2}$

$=\frac{\log_e 2}{1+\log_e 2}$

50. Question

Evaluate the following integrals

$$\int_{\pi/6}^{\pi/2}\frac{\text{cosec x cot x}}{1+\text{cosec}^2\text{x}}\text{dx}$$

Answer

Let $I=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{cosecx cotx}{1+cosec^2x}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{cosx}{1+sin^2x}dx$

Let sinx=t

⇒ cos x dx=dt.

Also, when $x=\frac{\pi}{6}$, $t=\frac{1}{2}$ and when $x=\frac{\pi}{2}$, t=1.

$$I=\int_{\frac{1}{2}}^{1}\frac{1}{1+t^2}dt$$

$$=tan^{-1}t\Big|_{\frac{1}{2}}^{1}$$

$$=tan^{-1}1-tan^{-1}\left(\frac{1}{2}\right)$$

$$=tan^{-1}\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)$$

$$=tan^{-1}\left(\frac{1}{3}\right)$$

(Using $tan^{-1}a-tan^{-1}b=tan^{-1}\left(\frac{a-b}{1+ab}\right)$)

Exercise 16C

1. Question

Prove that

$$\int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)} dx = \frac{\pi}{4}$$

Answer

$$y = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{2\cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos x + \cos x - \sin x + \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\left((x)_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\sin x + \cos x} dx \right)$$

Let, sin x + cos x = t

⇒ (cos x – sin x) dx = dt

At x = 0, t = 1

At x = π/2, t = 1

$$y = \frac{1}{2}\left(\frac{\pi}{2} + \int_1^1 \frac{1}{t} dt\right)$$

$$y = \frac{1}{2}(\frac{\pi}{2} + (\ln t)_1^1$$

$$y = \frac{\pi}{4}$$

2. Question

Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)} dx = \frac{\pi}{4}$$

Answer

$$y = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})} dx \ldots(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\left(\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}\right)} dx$$

$y = \int_0^{\pi/2} \frac{\sqrt{cos x}}{(\sqrt{cos x}+\sqrt{sin x})} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sqrt{sin\, x}}{(\sqrt{sin\, x}+\sqrt{cos\, x})} dx + \int_0^{\pi/2} \frac{\sqrt{cos\, x}}{(\sqrt{cos\, x}+\sqrt{sin\, x})} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{sin\, x}+\sqrt{\cos x}}{(\sqrt{sin\, x}+\sqrt{cos\, x})} dx$$

$$= \int_0^{\pi/2} 1\, dx$$

$$= (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

3 A. Question

Prove that

$$\int_0^{\pi/2} \frac{\sin^3 x}{\left(\sin^3 x + \cos^3 x\right)} dx = \frac{\pi}{4}$$

Answer

$y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{sin^3\left(\frac{\pi}{2}-x\right)}{sin^3\left(\frac{\pi}{2}-x\right)+cos^3\left(\frac{\pi}{2}-x\right)} dx$$

$y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x} dx + \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x} dx$$

$$= \int_0^{\pi/2} \frac{sin^3 x + cos^3 x}{sin^3 x + cos^3 x} dx$$

$$= \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

3 B. Question

Prove that

$$\int_0^{\pi/2} \frac{\cos^3 x\,dx}{(\sin^3 x + \cos^3 x)} = \frac{\pi}{4}$$

Answer

$y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{cos^3\left(\frac{\pi}{2} - x\right)}{sin^3\left(\frac{\pi}{2} - x\right) + cos^3\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x} dx + \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x} dx$$

$$2y = \int_0^{\pi/2} \frac{cos^3 x + sin^3 x}{sin^3 x + cos^3 x} dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

4 A. Question

Prove that

$$\int_0^{\pi/2} \frac{\sin^7 x}{(\sin^7 x + \cos^7 x)} dx = \frac{\pi}{4}$$

Answer

$y = \int_0^{\pi/2} \frac{sin^7 x}{sin^7 x + cos^7 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{sin^7\left(\frac{\pi}{2} - x\right)}{sin^7\left(\frac{\pi}{2} - x\right) + cos^7\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{cos^7 x}{sin^7 x + cos^7 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sin^7 x}{\sin^7 x + \cos^7 x} dx + \int_0^{\pi/2} \frac{\cos^7 x}{\sin^7 x + \cos^7 x} dx$$

$$2y = \int_0^{\pi/2} \frac{\sin^7 x + \cos^7 x}{\sin^7 x + \cos^7 x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

4 B. Question

Prove that

$$\int_0^{\pi/2} \frac{\cos^4 x}{(\sin^4 x + \cos^4 x)} dx = \frac{\pi}{4}$$

Answer

$y = \int_0^{\pi/2} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$$

$$y = \int_0^{\pi/2} \frac{\cos^4\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} dx + \int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$$

$$2y = \int_0^{\pi/2} \frac{\cos^4 x + \sin^4 x}{\sin^4 x + \cos^4 x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

5. Question

Prove that

$$\int_0^{\pi/2} \frac{\cos^4 x}{(\sin^4 x + \cos^4 x)} dx = \frac{\pi}{4}$$

Answer

$y = \int_0^{\pi/2} \frac{cos^4 x}{sin^4 x + cos^4 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{cos^4\left(\frac{\pi}{2} - x\right)}{sin^4\left(\frac{\pi}{2} - x\right) + cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{sin^4 x}{sin^4 x + cos^4 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cos^4 x}{sin^4 x + cos^4 x} dx + \int_0^{\pi/2} \frac{sin^4 x}{sin^4 x + cos^4 x} dx$$

$$2y = \int_0^{\pi/2} \frac{cos^4 x + sin^4 x}{sin^4 x + cos^4 x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

6. Question

Prove that

$$\int_0^{\pi/2} \frac{\cos^{1/4} x}{\left(\sin^{1/4} x + \cos^{1/4} x\right)} dx = \frac{\pi}{4}$$

Answer

$y = \int_0^{\pi/2} \frac{cos^{\frac{1}{4}} x}{sin^{\frac{1}{4}} x + cos^{\frac{1}{4}} x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{cos^{\frac{1}{4}}\left(\frac{\pi}{2} - x\right)}{sin^{\frac{1}{4}}\left(\frac{\pi}{2} - x\right) + cos^{\frac{1}{4}}\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{sin^{\frac{1}{4}} x}{sin^{\frac{1}{4}} x + cos^{\frac{1}{4}} x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cos^{\frac{1}{4}} x}{sin^{\frac{1}{4}} x + cos^{\frac{1}{4}} x} dx + \int_0^{\pi/2} \frac{sin^{\frac{1}{4}} x}{sin^{\frac{1}{4}} x + cos^{\frac{1}{4}} x} dx$$

$$2y = \int_0^{\pi/2} \frac{cos^{\frac{1}{4}}x + sin^{\frac{1}{4}}x}{sin^{\frac{1}{4}}x + cos^{\frac{1}{4}}x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

7. Question

Prove that

$$\int_0^{\pi/2} \frac{\sin^{3/2} x}{\left(\sin^{3/2} x + \cos^{3/2} x\right)} dx = \frac{\pi}{4}$$

Answer

$$y = \int_0^{\pi/2} \frac{sin^{\frac{3}{2}}x}{sin^{\frac{3}{2}}x + cos^{\frac{3}{2}}x} dx \ldots(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$$

$$y = \int_0^{\pi/2} \frac{sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}{sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)} dx$$

$$y = \int_0^{\pi/2} \frac{cos^{\frac{3}{2}}x}{sin^{\frac{3}{2}}x + cos^{\frac{3}{2}}x} dx \ldots(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{sin^{\frac{3}{2}}x}{sin^{\frac{3}{2}}x + cos^{\frac{3}{2}}x} dx + \int_0^{\pi/2} \frac{cos^{\frac{3}{2}}x}{sin^{\frac{3}{2}}x + cos^{\frac{3}{2}}x} dx$$

$$2y = \int_0^{\pi/2} \frac{sin^{\frac{3}{2}}x + cos^{\frac{3}{2}}x}{sin^{\frac{3}{2}}x + cos^{\frac{3}{2}}x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

8. Question

Prove that

$$\int_0^{\pi/2} \frac{\sin^n x}{\left(\sin^n x + \cos^n x\right)} dx = \frac{\pi}{4}$$

Answer

$y = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\sin^n\left(\frac{\pi}{2} - x\right)}{\sin^n\left(\frac{\pi}{2} - x\right) + \cos^n\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx + \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$$

$$2y = \int_0^{\pi/2} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

9. Question

Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\left(\sqrt{\tan x} + \sqrt{\cot x}\right)} dx = \frac{\pi}{4}$$

Answer

$$y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{\sin x}{\cos x}}}{\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}} dx$$

$y = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{sinx}{sinx + cosx}\,dx + \int_0^{\pi/2} \frac{cosx}{sinx + cosx}\,dx$$

$$2y = \int_0^{\pi/2} \frac{sinx + cosx}{sinx + cosx}\,dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

10. Question

Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\left(\sqrt{\tan x} + \sqrt{\cot x}\right)}\,dx = \frac{\pi}{4}$$

Answer

$$y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{\cos x}{\sin x}}}{\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}}\,dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx \quad \ldots(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\,dx$$

$$y = \int_0^{\pi/2} \frac{sinx}{sinx + cosx}\,dx \quad \ldots(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cosx}{sinx + cosx}\,dx + \int_0^{\pi/2} \frac{sinx}{sinx + cosx}\,dx$$

$$2y = \int_0^{\pi/2} \frac{sinx + cosx}{sinx + cosx}\,dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

11. Question

Prove that

$$\int_0^{\pi/2} \frac{dx}{(1+\tan x)} = \frac{\pi}{4}$$

Answer

$y = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\sin x}{\cos x}} dx$

$y = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} dx$$

$y = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

$$2y = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

12. Question

Prove that

$$\int_0^{\pi/2} \frac{dx}{(1+\cot x)} = \frac{\pi}{4}$$

Answer

$y = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\cos x}{\sin x}} dx$

$y = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} dx$$

$y = \int_0^{\pi/2} \frac{cosx}{sinx+cosx} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{sinx}{sinx + cosx} dx + \int_0^{\pi/2} \frac{cosx}{sinx + cosx} dx$$

$$2y = \int_0^{\pi/2} \frac{sinx + cosx}{sinx + cosx} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

13. Question

Prove that

$$\int_0^{\pi/2} \frac{dx}{(1+\tan^3 x)} = \frac{\pi}{4}$$

Answer

$y = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{sin^3 x}{cos^3 x}} dx$

$y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x+cos^3 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{cos^3\left(\frac{\pi}{2} - x\right)}{sin^3\left(\frac{\pi}{2} - x\right) + cos^3\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x+cos^3 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{cos^3 x}{sin^3 x + cos^3 x} dx + \int_0^{\pi/2} \frac{sin^3 x}{sin^3 x + cos^3 x} dx$$

$$2y = \int_0^{\pi/2} \frac{cos^3 x + sin^3 x}{sin^3 x + cos^3 x} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

14. Question

Prove that

$$\int_0^{\pi/2} \frac{dx}{\left(1+\cot^3 x\right)} = \frac{\pi}{4}$$

Answer

$$y = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\cos^3 x}{\sin^3 x}} dx$$

$$y = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x+\cos^3 x} dx \quad ...(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\sin^3\left(\frac{\pi}{2}-x\right)}{\sin^3\left(\frac{\pi}{2}-x\right)+\cos^3\left(\frac{\pi}{2}-x\right)} dx$$

$$y = \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x+\cos^3 x} dx \quad ...(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x+\cos^3 x} dx + \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x+\cos^3 x} dx$$

$$2y = \int_0^{\pi/2} \frac{\sin^3 x+\cos^3 x}{\sin^3 x+\cos^3 x} dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

15. Question

Prove that

$$\int_0^{\pi/2} \frac{dx}{\left(1+\sqrt{\tan x}\right)} = \frac{\pi}{4}$$

Answer

$$y = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt{\frac{\sin x}{\cos x}}} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)} dx \quad ...(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{cos\left(\frac{\pi}{2} - x\right)}}{\left(\sqrt{sin\left(\frac{\pi}{2} - x\right)} + \sqrt{cos\left(\frac{\pi}{2} - x\right)}\right)} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{sin\,x}}{(\sqrt{cos\,x} + \sqrt{sin\,x})} dx \ldots(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sqrt{cos\,x}}{(\sqrt{sin\,x} + \sqrt{cos\,x})} dx + \int_0^{\pi/2} \frac{\sqrt{sin\,x}}{(\sqrt{cos\,x} + \sqrt{sin\,x})} dx$$

$$2y = \int_0^{\pi/2} \frac{\sqrt{sin\,x} + \sqrt{cos\,x}}{(\sqrt{sin\,x} + \sqrt{cos\,x})} dx$$

$$2y = \int_0^{\pi/2} 1\,dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

16. Question

Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\left(1 + \sqrt{\cot x}\right)} dx = \frac{\pi}{4}$$

Answer

$$y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{\cos x}{\sin x}}}{1 + \sqrt{\frac{\cos x}{\sin x}}} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{cos x}}{(\sqrt{sin x} + \sqrt{cos x})} dx \ldots(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{cos\left(\frac{\pi}{2} - x\right)}}{\left(\sqrt{sin\left(\frac{\pi}{2} - x\right)} + \sqrt{cos\left(\frac{\pi}{2} - x\right)}\right)} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{sin\,x}}{(\sqrt{cos\,x} + \sqrt{sin\,x})} dx \ldots(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sqrt{cos\,x}}{(\sqrt{sin\,x} + \sqrt{cos\,x})} dx + \int_0^{\pi/2} \frac{\sqrt{sin\,x}}{(\sqrt{cos\,x} + \sqrt{sin\,x})} dx$$

$$2y = \int_0^{\pi/2} \frac{\sqrt{sin\,x} + \sqrt{cos\,x}}{(\sqrt{sin\,x} + \sqrt{cos\,x})} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

17. Question

Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\left(1+\sqrt{\tan x}\right)} dx = \frac{\pi}{4}$$

Answer

$$y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{\sin x}{\cos x}}}{1+\sqrt{\frac{\sin x}{\cos x}}} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx \ ...(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\left(\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}\right)} dx$$

$$y = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\cos x}+\sqrt{\sin x})} dx \ ...(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\cos x}+\sqrt{\sin x})} dx$$

$$2y = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{(\sqrt{\sin x}+\sqrt{\cos x})} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

18. Question

Prove that

$$\int_0^{\pi/2} \frac{(\sin x - \cos x)}{(1+\sin x \cos x)} dx = 0$$

Answer

$y = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)} dx$$

$y = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\cos x \sin x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\cos x \sin x} dx$$

$$2y = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x + \cos x - \sin x}{1+\cos x \sin x} dx$$

$$2y = \int_0^{\frac{\pi}{2}} 0\,dx$$

y = 0

19. Question

Prove that

$$\int_0^1 x(1-x)^5\,dx = \frac{1}{42}$$

Answer

$y = \int_0^1 x(1-x)^5\,dx$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^1 (1-x)x^5\,dx$$

$$y = \int_0^1 x^5 - x^6\,dx$$

$$y = \left(\frac{x^6}{6} - \frac{x^7}{7}\right)_0^1$$

$$y = \frac{1}{6} - \frac{1}{7}$$

$= \frac{1}{42}$

20. Question

Prove that

$$\int_0^2 x\sqrt{2-x}\, dx = \frac{16\sqrt{2}}{15}$$

Answer

$y = \int_0^2 x\sqrt{2-x}\, dx$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^2 (2-x)\sqrt{x}\, dx$$

$$y = \int_0^2 2x^{\frac{1}{2}} - x^{\frac{3}{2}}\, dx$$

$$y = \left(2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_0^2$$

$$y = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} = \frac{16\sqrt{2}}{15}$$

21. Question

Prove that

$$\int_0^\pi x\cos^2 x\, dx = \frac{\pi^2}{4}$$

Answer

$y = \int_0^\pi x\, cos^2 x\, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^\pi (\pi - x) cos^2(\pi - x)\, dx$$

$y = \int_0^\pi \pi\, cos^2 x - x\, cos^2 x\, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^\pi x\, cos^2 x\, dx + \int_0^\pi \pi\, cos^2 x - x\, cos^2 x\, dx$$

$$2y=\int_0^{\pi}\pi\cos^2 x\,dx$$

$$y=\frac{\pi}{2}\int_0^{\pi}\frac{1+\cos 2x}{2}dx$$

$$y=\frac{\pi}{2}\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)_0^{\pi}$$

$$y=\frac{\pi}{2}\left(\frac{\pi}{2}+\frac{\sin 2\pi}{4}\right)=\frac{\pi^2}{4}$$

22. Question

Prove that

$$\int_0^{\pi}\frac{x\tan x}{(\sec x\,\mathrm{cosec}\,x)}dx=\frac{\pi^2}{4}$$

Answer

$y=\int_0^{\pi}\frac{x\tan x}{\sec x\ cosec\ x}dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

$$y=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)\ cosec\ (\pi-x)}dx$$

$$y=\int_0^{\pi}\frac{-(\pi-x)\tan x}{-\sec x\ cosec\ x}dx$$

$y=\int_0^{\pi}\frac{\pi\tan x-x\tan x}{\sec x\ cosec\ x}dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y=\int_0^{\pi}\frac{x\tan x}{\sec x\ cosec\ x}dx+\int_0^{\pi}\frac{\pi\tan x-x\tan x}{\sec x\ cosec\ x}dx$$

$$2y=\int_0^{\pi}\frac{\pi\tan x}{\sec x\ cosec\ x}dx$$

$$y=\frac{\pi}{2}\int_0^{\pi}\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}\times\frac{1}{\sin x}}dx$$

$$y=\frac{\pi}{2}\int_0^{\pi}\frac{1-\cos 2x}{2}dx$$

$$y=\frac{\pi}{2}\left(\frac{x}{2}-\frac{\sin 2x}{4}\right)_0^{\pi}$$

$$y=\frac{\pi}{2}\left(\frac{\pi}{2}-\frac{\sin 2\pi}{4}\right)=\frac{\pi^2}{4}$$

23. Question

Prove that

$$\int_0^{\pi/2} \frac{\cos^2 x}{(\sin x+\cos x)} dx = \frac{1}{\sqrt{2}} \log\left(\sqrt{2}+1\right)$$

Answer

$y = \int_0^{\frac{\pi}{2}} \frac{cos^2 x}{\sin x+\cos x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{cos^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} dx$$

$y = \int_0^{\frac{\pi}{2}} \frac{sin^2 x}{\sin x+\cos x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\frac{\pi}{2}} \frac{cos^2 x}{\sin x+\cos x} dx + \int_0^{\frac{\pi}{2}} \frac{sin^2 x}{\sin x+\cos x} dx$$

$$2y = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x+\cos x} dx$$

$$2y = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x} dx$$

$$2y = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\sin\left(x+\frac{\pi}{4}\right)} dx$$

$$y = \frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} cosec\,\left(x+\frac{\pi}{4}\right) dx$$

$$y = \frac{1}{2\sqrt{2}}\left(\ln\left(\text{cosec}\left(x+\frac{\pi}{4}\right)-\cot\left(x+\frac{\pi}{4}\right)\right)\right)_0^{\frac{\pi}{2}}$$

$$y = \frac{1}{2\sqrt{2}}\left(\ln\left(\text{cosec}\frac{3\pi}{4}-\cot\frac{3\pi}{4}\right)-\ln\left(\text{cosec}\frac{\pi}{4}-\cot\frac{\pi}{4}\right)\right)$$

$$y = \frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$y = \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)^2 = \frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)$$

24. Question

Prove that

$$\int_0^{\pi} \frac{x \tan x}{(\sec x + \cos x)} dx = \frac{\pi^2}{4}$$

Answer

$y = \int_0^{\pi} \frac{x \frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \cos x} dx$

$y = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$$

$$y = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

$y = \int_0^{\pi} \frac{\pi \sin x - x \sin x}{1 + \cos^2 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{\pi \sin x - x \sin x}{1 + \cos^2 x} dx$$

$$2y = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx$$

Let, cos x = t

⇒ -sin x dx = dt

At x = 0, t = 1

At x = π, t = -1

$$y = -\frac{\pi}{2} \int_1^{-1} \frac{1}{1 + t^2} dt$$

$$y = -\frac{\pi}{2} (\tan^{-1} t)_1^{-1}$$

$$y = -\frac{\pi}{2} (\tan^{-1}(-1) - \tan^{-1} 1)$$

$$y = \frac{\pi^2}{4}$$

25. Question

Prove that

$$\int_0^{\pi} \frac{x \sin x}{(1 + \sin x)} dx = \pi\left(\frac{\pi}{2} - 1\right)$$

Answer

$y = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)}\,dx$$

$$y = \int_0^{\pi} \frac{\pi \sin x}{1+\sin x} - \frac{x\sin x}{1+\sin x}\,dx \ldots(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx + \int_0^{\pi} \frac{\pi \sin x}{1+\sin x} - \frac{x\sin x}{1+\sin x}\,dx$$

$$2y = \int_0^{\pi} \frac{\pi(\sin x + 1 - 1)}{1+\sin x}\,dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} 1 - \frac{1}{1+\sin x}\,dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} 1 - \frac{1-\sin x}{cos^2 x}\,dx$$

$$y = \frac{\pi}{2}\int_0^{\pi} 1 - sec^2 x + \frac{\sin x}{cos^2 x}\,dx$$

Let, cos x = t

⇒ -sin x dx = dt

At x = 0, t = 1

At x = π, t = -1

$$y = \frac{\pi}{2}\left((x - \tan x)_0^{\pi} - \int_1^{-1} \frac{1}{t^2}\,dt\right)$$

$$y = \frac{\pi}{2}\left(\pi - \tan \pi - \left(\frac{-1}{t}\right)_1^{-1}\right)$$

$$y = \frac{\pi}{2}(\pi - 2) = \pi\left(\frac{\pi}{2} - 1\right)$$

26. Question

Prove that

$$\int_0^{\pi} \frac{\mathrm{x}}{\left(1+\sin^2 \mathrm{x}\right)}\,\mathrm{dx} = \frac{\pi^2}{2\sqrt{2}}$$

Answer

$$y = \int_0^{\pi} \frac{x}{1+sin^2 x}\,dx \ldots(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^\pi \frac{(\pi - x)}{1 + sin^2(\pi - x)} dx$$

$y = \int_0^\pi \frac{\pi}{1+sin^2 x} - \frac{x}{1+sin^2 x} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^\pi \frac{x}{1 + sin^2 x} dx + \int_0^\pi \frac{\pi}{1 + sin^2 x} - \frac{x}{1 + sin^2 x} dx$$

$$y = \frac{\pi}{2}\int_0^\pi \frac{1}{1 + sin^2 x} dx$$

$$y = \frac{\pi}{2}\int_0^\pi \frac{\frac{1}{cos^2 x}}{\frac{1 + sin^2 x}{cos^2 x}} dx$$

$$y = \frac{\pi}{2}\int_0^\pi \frac{sec^2 x}{sec^2 x + tan^2 x} dx$$

We break it in two parts

$$y = \frac{\pi}{2}\int_0^\pi \frac{sec^2 x}{sec^2 x + tan^2 x} dx$$

Let, tan x = t

$\Rightarrow sec^2 x\, dx = dt$

At x = 0, t = 0

At x = π, t = 0

$$y = \frac{\pi}{2}\int_0^0 \frac{1}{1 + 2t^2} dt$$

We know that when upper and lower limit is same in definite

integral then value of integration is 0.

So, y = 0

27. Question

Prove that

$$\int_0^{\pi/2} (2\log\cos x - \log\sin 2x)\,dx = -\frac{\pi}{4}(\log 2)$$

Answer

$y = \int_0^{\frac{\pi}{2}} \log\frac{cos^2 x}{\sin 2x} dx$

$$y = \int_0^{\frac{\pi}{2}} \log\frac{cos^2 x}{2\sin x\cos x}dx$$

$y = \int_0^{\frac{\pi}{2}}\log\left(\frac{1}{2}\cot x\right)dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\cot\left(\frac{\pi}{2}-x\right)\right)dx$$

$y = \int_0^{\frac{\pi}{2}}\log\left(\frac{1}{2}\tan x\right)dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\cot x\right)dx + \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\tan x\right)dx$$

$y = \frac{1}{2}\int_0^{\frac{\pi}{2}}\log\left(\frac{1}{4}\cot x\tan x\right)dx$ [Use cot x tan x = 1]

$$y = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left(\frac{1}{4}\right)dx$$

$$y = \frac{1}{2}\log\left(\frac{1}{4}\right)(x)_0^{\frac{\pi}{2}}$$

$$y = -\frac{\pi}{4}\log 4$$

28. Question

Prove that

$$\int_0^{\infty}\frac{x}{(1+x)(1+x^2)}dx = \frac{\pi}{4}$$

Answer

$$y = \int_0^{\infty}\frac{x}{(1+x)(1+x^2)}dx$$

Let, x = tan t

⇒ dx = sec^2t dt

At x = 0, t = 0

At x = ∞, t = π/2

$$y = \int_0^{\frac{\pi}{2}} \frac{\tan t}{(1+\tan t)(1+tan^2 t)}sec^2 t\,dt$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\tan t}{(1+\tan t)}\, dt$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\sin t}{(\cos t+\sin t)}\, dt \;...(1)$$

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$y = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-t\right)}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)} dt$$

$$y = \int_0^{\pi/2} \frac{\cos t}{\sin t+\cos t} dt \;...(2)$$

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi/2} \frac{\sin t}{\sin t+\cos t} dx + \int_0^{\pi/2} \frac{\cos t}{\sin t+\cos t} dx$$

$$2y = \int_0^{\pi/2} \frac{\sin t+\cos t}{\sin t+\cos t} dx$$

$$2y = \int_0^{\pi/2} 1\, dx$$

$$2y = (x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

29. Question

Prove that

$$\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$$

Answer

Let, x = a sin t

⇒ dx = a cos t dt

At x = 0, t = 0

At x = a, t = π/2

$$y = \int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\sin t+\sqrt{a^2-a^2\sin^2 t}} dt$$

$$y = \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t} dt$$

$$y = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos t + \cos t - \sin t + \sin t}{\sin t + \cos t} dt$$

$$y = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \frac{\cos t - \sin t}{\sin t + \cos t} dt$$

$$y = \frac{1}{2}\left((t)_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos t - \sin t}{\sin t + \cos t} dt \right)$$

Again, sin t + cos t = z

⇒ (cos t – sin t) dt = dz

At t = 0, z = 1

At t = π/2, z = 1

$$y = \frac{1}{2}\left(\frac{\pi}{2} + \int_1^1 \frac{1}{z} dz \right)$$

$$y = \frac{1}{2}(\frac{\pi}{2} + (\ln z)_1^1$$

$$y = \frac{\pi}{4}$$

30. Question

$$\int_0^a \frac{\sqrt{x}}{\left(\sqrt{x} + \sqrt{a - x}\right)} dx = \frac{\pi}{4}$$

Answer

$y = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$$

$y = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx + \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} dx$$

$$2y = \int_0^a \frac{\sqrt{x} + \sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} dx$$

$$y = \frac{1}{2}\int_0^a dx$$

$y = \frac{1}{2}(x)_0^a$

$y = \frac{a}{2}$

31. Question

Prove that

$$\int_0^{\pi} \sin^2 x \cos^3 x \, dx = 0$$

Answer

$y = \int_0^{\pi} sin^2 x \, cos^3 x \, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} sin^2(\pi - x)\, cos^3(\pi - x)\, dx$$

$y = -\int_0^{\pi} sin^2 x \, cos^3 x \, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} sin^2 x \, cos^3 x \, dx + \left(-\int_0^{\pi} sin^2 x \, cos^3 x \, dx\right)$$

y = 0

32. Question

Prove that

$\int_0^{\pi} \sin^{2m} x \cos^{2m+1} x \, dx = 0$, where m is a positive integer

Answer

$y = \int_0^{\pi} sin^{2m} x \, cos^{2m+1} x \, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$y = \int_0^{\pi} sin^{2m}(\pi - x)\, cos^{2m+1}(\pi - x)\, dx$$

$y = -\int_0^{\pi} sin^{2m} x \, cos^{2m+1} x \, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2y = \int_0^{\pi} sin^{2m} x \, cos^{2m+1} x \, dx + \left(-\int_0^{\pi} sin^{2m} x \, cos^{2m+1} x \, dx\right)$$

y = 0

33. Question

Prove that

$$\int_0^{\pi/2} (\sin x - \cos x)\log(\sin x + \cos x)\,dx = 0$$

Answer

Let, sin x + cos x = t

⇒ cos x - sin x dx = dt

At x = 0, t = 1

At x = π/2, t = 1

$$y = \int_1^1 -\log t\, dt$$

We know that when upper and lower limit in definite integral is

equal then value of integration is zero.

So, y = 0

34. Question

Prove that

$$\int_0^{\pi/2} \log(\sin 2x)\,dx = -\frac{\pi}{2}(\log 2)$$

Answer

$$y = \int_0^{\frac{\pi}{2}} \log(2 \sin x \cos x)\, dx$$

$$y = \int_0^{\frac{\pi}{2}} \log 2 + \log\sin x + \log\cos x\, dx$$

Let, $I = \int_0^{\frac{\pi}{2}} \log\sin x\, dx$ ...(1)

Use King theorem of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right) dx$$

$I = \int_0^{\frac{\pi}{2}} \log\cos x\, dx$ ...(2)

Adding eq.(1) and eq.(2)

$$2I = \int_0^{\frac{\pi}{2}} \log\sin x\, dx + \int_0^{\frac{\pi}{2}} \log\cos x\, dx$$