EXERCISE 19.18

Evaluate the following integrals:

1. $\int \frac{x}{\sqrt{x^4 + a^4}} dx$

Solution:

The given equation can be written as

$$\int \frac{x}{\sqrt{x^4 + a^4}} dx = \int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx$$

Let $x^2 = t$, so $2x\, dx = dt$

Or, $x\, dx = dt/2$

Hence, $\int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx = \int \frac{1}{\sqrt{t^2 + (a^2)^2}} \frac{dt}{2} = \frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt$

Since, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$

Hence, $\frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt = \frac{1}{2} \log|t + \sqrt{t^2 + (a^2)^2}| + c$

Put $t = x^2$

$= \frac{1}{2} \log| x^2 + \sqrt{(x^2)^2 + (a^2)^2}| + c$

$= \frac{1}{2} \log| x^2 + \sqrt{x^4 + a^4}| + c$

2. $\int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx$

Solution:

Let tan x = t

Then $dt = \sec^2 x\, dx$

Therefore, $\int \frac{\sec^2 x}{\sqrt{4+\tan^2 x}} dx = \int \frac{dt}{\sqrt{2^2+t^2}}$

Since, $\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$

Hence, $\int \frac{dt}{\sqrt{2^2+t^2}} = \log|t + \sqrt{t^2 + 2^2}| + c$

$= \log|\tan x + \sqrt{\tan^2 x + 4}| + c$

3. $\int \frac{e^x}{\sqrt{16 - e^{2x}}} dx$

Solution:

Let $e^x = t$

Then we have, $e^x\, dx = dt$

Substituting these values,

Therefore, $\int \frac{e^x}{\sqrt{16-e^{2x}}} dx = \int \frac{dt}{\sqrt{4^2-t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{dt}{\sqrt{4^2-t^2}} = \sin^{-1}\left(\frac{e^x}{4}\right) + c$

4. $\int \frac{\cos x}{\sqrt{4 + \sin^2 x}} dx$

Solution:

Let sin x = t

Then $dt = \cos x\, dx$

Now substituting these values we get

Hence, $\int \frac{\cos x}{\sqrt{4+\sin^2 x}} dx = \int \frac{dt}{\sqrt{2^2+t^2}}$

Since we have, $\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

Therefore, $\int \frac{dt}{\sqrt{2^2+t^2}} = \log\left| t + \sqrt{t^2 + 2^2} \right| + c$

$= \log\left| t + \sqrt{t^2 + 2^2} \right| + c = \log\left| \sin x + \sqrt{\sin^2 x + 4} \right| + c$

5. $\int \frac{\sin x}{\sqrt{4\cos^2 x - 1}} dx$

Solution:

Let

$2\cos x = t$

Then $dt = -2\sin x\, dx$

Or, $\sin x\, dx = -\frac{dt}{2}$

Then substituting these values we get,

Therefore, $\int \frac{\sin x}{\sqrt{4\cos^2 x - 1}} dx = \int -\frac{dt}{2\sqrt{(t^2-1^2)}}$

Since, $\int \frac{1}{\sqrt{(x^2-a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

Therefore, $\int -\frac{dt}{2\sqrt{(t^2-1^2)}} = -\frac{1}{2}\log\left|t + \sqrt{t^2-1}\right| + c$

On integrating we get

$= -\frac{1}{2}\log\left|2\cos x + \sqrt{4\cos^2 x - 1}\right| + c$

6. $\int \frac{x}{\sqrt{4-x^4}}\, dx$

Solution:

Let $x^2 = t$

$2x\, dx = dt$ or $x\, dx = dt/2$

Now substituting these values in the given equation we get

Hence, $\int \frac{x}{\sqrt{4-x^4}} dx = \int \frac{dt}{2\left(\sqrt{2^2-t^2}\right)}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

So, $\int \frac{dt}{2\left(\sqrt{2^2-t^2}\right)} = \frac{1}{2}\sin^{-1}\left(\frac{t}{2}\right) + c$

Put $t = x^2$

$= \frac{1}{2}\sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{2}\right) + c$

7. $\int \frac{1}{x\sqrt{4-9(\log x)^2}}\, dx$

Solution:

Let 3 log x = t

We have $d(\log x) = 1/x$

Hence, $d(3\log x) = dt = 3/x\, dx$

Or $1/x\, dx = dt/3$

Hence, $\int \frac{1}{x\sqrt{4-9(\log x)^2}} dx = \int \frac{1}{3}\frac{dt}{\sqrt{2^2-t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{1}{3}\frac{dt}{\sqrt{2^2-t^2}} = \frac{1}{3}\sin^{-1}\left(\frac{t}{2}\right) + c$

Put t = 3 log x

$= \frac{1}{3}\sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{3}\sin^{-1}\left(\frac{3\log x}{2}\right) + c$

8. $\int \frac{\sin 8x}{\sqrt{9+\sin^4 4x}}\,dx$

Solution:

dt = 2sin 4x cos 4x × 4 dx

We know sin 2x = 2 sin 2x cos 2x

Therefore, dt = 4 sin 8x dx

Or, sin 8x dx = dt/4

$\int \frac{\sin 8x}{\sqrt{9+\sin^4 4x}}\,dx = \frac{1}{4}\int \frac{dt}{\sqrt{3^2+t^2}}$

Since we have, $\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \log|x + \sqrt{(x^2+a^2)}| + c$

$= \frac{1}{4}\int \frac{dt}{\sqrt{3^2+t^2}} = \frac{1}{4}\log|t + \sqrt{t^2+3^2}| + c$

$= \frac{1}{4}\log[\sin^2 4x + \sqrt{9+\sin^4 4x} + c$

9. $\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}}\,dx$

Solution:

Let t = sin2x

dt = 2 cos 2x dx

Cos 2x dx = dt/2

$\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}} dx = \frac{1}{2} \int dt/\sqrt{(t^2 + (2\sqrt{2})^2}$

Since we have, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$

$= \frac{1}{2} \int dt/\sqrt{(t^2 + (2\sqrt{2})^2} = \frac{1}{2}\log\left|t + \sqrt{t^2 + 8}\right| + c$

$= \frac{1}{2}\log|t + \sqrt{t^2 + 8}| + c = \frac{1}{2}\log\left|\sin 2x + \sqrt{\sin^2 2x + 8}\right| + c$