EXERCISE 16.2

1. Find the equation of the tangent to the curve √x + √y = a, at the point ($a^2/4$, $a^2/4$).

Solution:

Given √x + √y = a

To find the slope of the tangent of the given curve we have to differentiate the given equation

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{x}}{\sqrt{y}}$$

At $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$ slope m, is – 1

The equation of the tangent is given by $y - y_1 = m(x - x_1)$

$$y - \frac{a^2}{4} = -1\left(x - \frac{a^2}{4}\right)$$

$$x + y = \frac{a^2}{2}$$

2. Find the equation of the normal to y = $2x^3 - x^2 + 3$ at (1, 4).

Solution:

Given $y = 2x^3 - x^2 + 3$

By differentiating the given curve, we get the slope of the tangent

$$m = \frac{dy}{dx} = 6x^2 - 2x$$

m = 4 at (1, 4)

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

$$m(\text{normal}) = -\frac{1}{4}$$

Equation of normal is given by $y - y_1 = m\ (\text{normal})\ (x - x_1)$

$$y - 4 = \left(-\frac{1}{4}\right)(x - 1)$$

$x + 4y = 17$

3. Find the equation of the tangent and the normal to the following curves at the indicated points:
(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

Solution:
Given $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

m (tangent) at (0, 5) = – 10

$$m(\text{normal}) \text{ at } (0,5) = \frac{1}{10}$$

Equation of tangent is given by $y - y_1 = m\ (\text{tangent})\ (x - x_1)$

$y - 5 = -10x$

$y + 10x = 5$

Equation of normal is given by $y - y_1 = m\ (\text{normal})\ (x - x_1)$

$$y - 5 = \frac{1}{10}x$$

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at x = 1 y = 3

Solution:

Given $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $x = 1$ $y = 3$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

m (tangent) at (x = 1) = 2

Normal is perpendicular to tangent so, $m_1m_2 = -1$

$$m(\text{normal}) \text{ at } (x = 1) = -\frac{1}{2}$$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$y - 3 = 2(x - 1)$

$y = 2x + 1$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$2y = 7 - x$$

(iii) $y = x^2$ at (0, 0)

Solution:
Given $y = x^2$ at (0, 0)
By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 2x$$

m (tangent) at (x = 0) = 0
Normal is perpendicular to tangent so, $m_1m_2 = -1$

$$m(\text{normal}) \text{ at } (x = 0) = \frac{1}{0}$$

We can see that the slope of normal is not defined
Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$
$y = 0$
Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

x = 0

(iv) $y = 2x^2 - 3x - 1$ at (1, – 2)

Solution:

Given $y = 2x^2 - 3x - 1$ at (1, – 2)

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 4x - 3$$

m (tangent) at (1, – 2) = 1

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at (1, – 2) = – 1

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

y + 2 = 1(x – 1)

y = x – 3

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

y + 2 = – 1(x – 1)

y + x + 1 = 0

$$(v)\ y^2 = \frac{x^3}{4-x}$$

Solution:

By differentiating the given curve, we get the slope of the tangent

$$2y\frac{dy}{dx} = \frac{(4-x)3x^2 + x^4}{(4-x)^2}$$

$$\frac{dy}{dx} = \frac{(4-x)3x^2 + x^4}{2y(4-x)^2}$$

m (tangent) at (2, – 2) = – 2

$$\text{m(normal) at } (2, -2) = \frac{1}{2}$$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

y + 2 = – 2(x – 2)

y + 2x = 2

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y + 2 = \frac{1}{2}(x - 2)$$

$2y + 4 = x - 2$

$2y - x + 6 = 0$

4. Find the equation of the tangent to the curve x = θ + sin θ, y = 1 + cos θ at θ = π/4.

Solution:

Given $x = \theta + \sin\theta$, $y = 1 + \cos\theta$ at $\theta = \pi/4$

By differentiating the given equation with respect to θ, we get the slope of the tangent

$$\frac{dx}{d\theta} = 1 + \cos\theta$$

$$\frac{dy}{d\theta} = -\sin\theta$$

Dividing both the above equations

$$\frac{dy}{dx} = -\frac{\sin\theta}{1 + \cos\theta}$$

m at θ = ($\pi/4$) = $-1 + \frac{1}{\sqrt{2}}$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - 1 - \frac{1}{\sqrt{2}} = \left(-1 + \frac{1}{\sqrt{2}}\right)\left(x - \frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$$

5. Find the equation of the tangent and the normal to the following curves at the indicated points:

(i) x = θ + sin θ, y = 1 + cos θ at θ = π/2

Solution:

Given $x = \theta + \sin\theta$, $y = 1 + \cos\theta$ at $\theta = \pi/2$

By differentiating the given equation with respect to θ, we get the slope of the tangent

$$\frac{dx}{d\theta} = 1 + \cos\theta$$

$$\frac{dy}{d\theta} = -\sin\theta$$

Dividing both the above equations

$$\frac{dy}{dx} = -\frac{\sin\theta}{1 + \cos\theta}$$

m (tangent) at θ = (π/2) = – 1

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at θ = (π/2) = 1

Equation of tangent is given by $y - y_1 = m \text{ (tangent) } (x - x_1)$

$$y - 1 = -1\left(x - \frac{\pi}{2} - 1\right)$$

Equation of normal is given by $y - y_1 = m \text{ (normal) } (x - x_1)$

$$y - 1 = 1\left(x - \frac{\pi}{2} - 1\right)$$

$$(ii)\ x = \frac{2at^2}{1+t^2},\ y = \frac{2at^3}{1+t^2}\ at\ t = \frac{1}{2}$$

Solution:

By differentiating the given equation with respect to t, we get the slope of the tangent

$$\frac{dx}{dt} = \frac{(1 + t^2)4at - 2at^2(2t)}{(1 + t^2)^2}$$

$$\frac{dx}{dt} = \frac{4at}{(1 + t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1 + t^2)6at^2 - 2at^3(2t)}{(1 + t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at^2 + 2at^4}{(1 + t^2)^2}$$

Now dividing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{6at^2 + 2at^4}{4at}$$

m (tangent) at t = $\frac{1}{2}$ is $\frac{13}{16}$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at t = $\frac{1}{2}$ is $-\frac{16}{13}$

Equation of tangent is given by $y - y_1$ = m (tangent) $(x - x_1)$

$$y - \frac{a}{5} = \frac{13}{16}\left(x - \frac{2a}{5}\right)$$

Equation of normal is given by $y - y_1$ = m (normal) $(x - x_1)$

$$y - \frac{a}{5} = -\frac{16}{13}\left(x - \frac{2a}{5}\right)$$

(iii) x = at^2, y = 2at at t = 1.

Solution:

Given x = at^2, y = 2at at t = 1.

By differentiating the given equation with respect to t, we get the slope of the tangent

$$\frac{dx}{dt} = 2at$$

$$\frac{dy}{dt} = 2a$$

Now dividing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{1}{t}$$

m (tangent) at t = 1 is 1

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at t = 1 is – 1

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

y – 2a = 1(x – a)

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

y – 2a = – 1(x – a)

(iv) x = a sec t, y = b tan t at t.

Solution:

Given x = a sec t, y = b tan t at t.

By differentiating the given equation with respect to t, we get the slope of the tangent

$$\frac{dx}{dt} = a\sec t\tan t$$

$$\frac{dy}{dt} = b\sec^2 t$$

Now dividing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{b\operatorname{cosec} t}{a}$$

m (tangent) at t = $\frac{b\operatorname{cosec} t}{a}$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at t = $-\frac{a}{b}\sin t$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - b\tan t = \frac{b\operatorname{cosec} t}{a}(x - a\sec t)$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - b\tan t = -\frac{a\sin t}{b}(x - a\sec t)$$

(v) x = a (θ + sin θ), y = a (1 – cos θ) at θ

Solution:

Given $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at θ

By differentiating the given equation with respect to θ, we get the slope of the tangent

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\frac{dy}{d\theta} = a(\sin\theta)$$

Now dividing $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{\sin\theta}{1 + \cos\theta}$$

m (tangent) at theta is $\frac{\sin\theta}{1 + \cos\theta}$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at theta is $-\frac{\sin\theta}{1 + \cos\theta}$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - a(1 - \cos\theta) = \frac{\sin\theta}{1 + \cos\theta}(x - a(\theta + \sin\theta))$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - a(1 - \cos\theta) = \frac{1 + \cos\theta}{-\sin\theta}(x - a(\theta + \sin\theta))$$

(vi) $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$

Solution:

Given $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$

By differentiating the given equation with respect to θ, we get the slope of the tangent

$$\frac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$$

Now dividing $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{3\cos\theta - 3\sin^2\theta\cos\theta}{-3\sin\theta + 3\cos^2\theta\sin\theta} = -\tan^3\theta$$

m (tangent) at theta is $-\tan^3\theta$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at theta is $\cot^3\theta$

Equation of tangent is given by $y - y_1 = m\text{ (tangent) }(x - x_1)$

$$y - 3\sin\theta + \sin^3\theta = -\tan^3\theta(x - 3\cos\theta + 3\cos^3\theta)$$

Equation of normal is given by $y - y_1 = m\text{ (normal) }(x - x_1)$

$$y - 3\sin\theta + \sin^3\theta = \cot^3\theta(x - 3\cos\theta + 3\cos^3\theta)$$

6. Find the equation of the normal to the curve $x^2 + 2y^2 - 4x - 6y + 8 = 0$ at the point whose abscissa is 2.

Solution:

Given $x^2 + 2y^2 - 4x - 6y + 8 = 0$

By differentiating the given curve, we get the slope of the tangent

$$2x + 4y\frac{dy}{dx} - 4 - 6\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{4 - 2x}{4y - 6}$$

Finding y co – ordinate by substituting x in the given curve

$2y^2 - 6y + 4 = 0$

$y^2 - 3y + 2 = 0$

y = 2 or y = 1

m (tangent) at x = 2 is 0

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at x = 2 is 1/0, which is undefined

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

x = 2

7. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3).

Solution:

Given $ay^2 = x^3$

By differentiating the given curve, we get the slope of the tangent

$$2ay\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$

m (tangent) at (am^2, am^3) is $\frac{3m}{2}$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) at (am^2, am^3) is $-\frac{2}{3m}$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - am^3 = -\frac{2}{3m}(x - am^2)$$

8. The equation of the tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of a and b.

Solution:

Given $y^2 = ax^3 + b$ is $y = 4x - 5$

By differentiating the given curve, we get the slope of the tangent

$$2y\frac{dy}{dx} = 3ax^2$$

$$\frac{dy}{dx} = \frac{3ax^2}{2y}$$

m (tangent) at (2, 3) = 2a

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

Now comparing the slope of a tangent with the given equation

2a = 4

a = 2

Now (2, 3) lies on the curve, these points must satisfy

$3^2 = 2 \times 2^3 + b$

b = – 7

9. Find the equation of the tangent line to the curve $y = x^2 + 4x - 16$ which is parallel to the line $3x - y + 1 = 0$.

Solution:

Given $y = x^2 + 4x - 16$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 2x + 4$$

m (tangent) = 2x + 4

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

Now comparing the slope of a tangent with the given equation

2x + 4 = 3

$$x = -\frac{1}{2}$$

Now substituting the value of x in the curve to find y

$$y = \frac{1}{4} - 2 - 16 = -\frac{71}{4}$$

Therefore, the equation of tangent parallel to the given line is

$$y + \frac{71}{4} = 3\left(x + \frac{1}{2}\right)$$

10. Find the equation of normal line to the curve $y = x^3 + 2x + 6$ which is parallel to the line x + 14y + 4 = 0.

Solution:

Given $y = x^3 + 2x + 6$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 3x^2 + 2$$

m (tangent) = $3x^2 + 2$

Normal is perpendicular to tangent so, $m_1m_2 = -1$

m (normal) = $\frac{-1}{3x^2+2}$

Equation of normal is given by $y - y_1$ = m (normal) $(x - x_1)$

Now comparing the slope of normal with the given equation

m (normal) = $-\frac{1}{14}$

$$-\frac{1}{14} = -\frac{1}{3x^2 + 2}$$

x = 2 or – 2

Hence the corresponding value of y is 18 or – 6

So, equations of normal are

$$y - 18 = -\frac{1}{14}(x - 2)$$

Or

$$y + 6 = -\frac{1}{14}(x + 2)$$