

EXERCISE 3.3

1. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = 1/x$

(ii) $f(x) = 1/(x-7)$

(iii) $f(x) = (3x-2)/(x+1)$

(iv) $f(x) = (2x+1)/(x^2-9)$

(v) $f(x) = (x^2+2x+1)/(x^2-8x+12)$

Solution:

(i) $f(x) = 1/x$

We know, $f(x)$ is defined for all real values of x , except for the case when $x = 0$.

\therefore Domain of $f = \mathbb{R} - \{0\}$

(ii) $f(x) = 1/(x-7)$

We know, $f(x)$ is defined for all real values of x , except for the case when $x - 7 = 0$ or $x = 7$.

\therefore Domain of $f = \mathbb{R} - \{7\}$

(iii) $f(x) = (3x-2)/(x+1)$

We know, $f(x)$ is defined for all real values of x , except for the case when $x + 1 = 0$ or $x = -1$.

\therefore Domain of $f = \mathbb{R} - \{-1\}$

(iv) $f(x) = (2x+1)/(x^2-9)$

We know, $f(x)$ is defined for all real values of x , except for the case when $x^2 - 9 = 0$.

$$x^2 - 9 = 0$$

$$x^2 - 3^2 = 0$$

$$(x + 3)(x - 3) = 0$$

$$x + 3 = 0 \text{ or } x - 3 = 0$$

$$x = \pm 3$$

\therefore Domain of $f = \mathbb{R} - \{-3, 3\}$

(v) $f(x) = (x^2+2x+1)/(x^2-8x+12)$

We know, $f(x)$ is defined for all real values of x , except for the case when $x^2 - 8x + 12 = 0$.

$$x^2 - 8x + 12 = 0$$

$$x^2 - 2x - 6x + 12 = 0$$

$$x(x - 2) - 6(x - 2) = 0$$

$$(x - 2)(x - 6) = 0$$

$$x - 2 = 0 \text{ or } x - 6 = 0$$

$$x = 2 \text{ or } 6$$

$$\therefore \text{Domain of } f = \mathbb{R} - \{2, 6\}$$

2. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = \sqrt{(x-2)}$

(ii) $f(x) = 1/(\sqrt{(x^2-1)})$

(iii) $f(x) = \sqrt{(9-x^2)}$

(iv) $f(x) = \sqrt{(x-2)/(3-x)}$

Solution:

(i) $f(x) = \sqrt{(x-2)}$

We know the square of a real number is never negative.

$f(x)$ takes real values only when $x - 2 \geq 0$

$$x \geq 2$$

$$\therefore x \in [2, \infty)$$

$$\therefore \text{Domain } (f) = [2, \infty)$$

(ii) $f(x) = 1/(\sqrt{(x^2-1)})$

We know the square of a real number is never negative.

$f(x)$ takes real values only when $x^2 - 1 \geq 0$

$$x^2 - 1^2 \geq 0$$

$$(x + 1)(x - 1) \geq 0$$

$$x \leq -1 \text{ or } x \geq 1$$

$$\therefore x \in (-\infty, -1] \cup [1, \infty)$$

In addition, $f(x)$ is also undefined when $x^2 - 1 = 0$ because denominator will be zero and the result will be indeterminate.

$$x^2 - 1 = 0 \Rightarrow x = \pm 1$$

$$\text{So, } x \in (-\infty, -1] \cup [1, \infty) - \{-1, 1\}$$

$$x \in (-\infty, -1) \cup (1, \infty)$$

$$\therefore \text{Domain } (f) = (-\infty, -1) \cup (1, \infty)$$

(iii) $f(x) = \sqrt{(9-x^2)}$

We know the square of a real number is never negative.

$f(x)$ takes real values only when $9 - x^2 \geq 0$

$$9 \geq x^2$$

$$x^2 \leq 9$$

$$x^2 - 9 \leq 0$$

$$x^2 - 3^2 \leq 0$$

$$(x + 3)(x - 3) \leq 0$$

$$x \geq -3 \text{ and } x \leq 3$$

$$x \in [-3, 3]$$

$$\therefore \text{Domain (f)} = [-3, 3]$$

$$\text{(iv) } f(x) = \sqrt{(x-2)/(3-x)}$$

We know the square root of a real number is never negative.

$f(x)$ takes real values only when $x - 2$ and $3 - x$ are both positive and negative.

(a) Both $x - 2$ and $3 - x$ are positive

$$x - 2 \geq 0$$

$$x \geq 2$$

$$3 - x \geq 0$$

$$x \leq 3$$

Hence, $x \geq 2$ and $x \leq 3$

$$\therefore x \in [2, 3]$$

(b) Both $x - 2$ and $3 - x$ are negative

$$x - 2 \leq 0$$

$$x \leq 2$$

$$3 - x \leq 0$$

$$x \geq 3$$

Hence, $x \leq 2$ and $x \geq 3$

However, the intersection of these sets is null set. Thus, this case is not possible.

Hence, $x \in [2, 3] - [3]$

$$x \in [2, 3]$$

$$\therefore \text{Domain (f)} = [2, 3]$$

3. Find the domain and range of each of the following real valued functions:

(i) $f(x) = (ax+b)/(bx-a)$

(ii) $f(x) = (ax-b)/(cx-d)$

(iii) $f(x) = \sqrt{(x-1)}$

(iv) $f(x) = \sqrt{(x-3)}$

(v) $f(x) = (x-2)/(2-x)$

(vi) $f(x) = |x-1|$

(vii) $f(x) = -|x|$

(viii) $f(x) = \sqrt{(9-x^2)}$

Solution:

(i) $f(x) = (ax+b)/(bx-a)$

$f(x)$ is defined for all real values of x , except for the case when $bx - a = 0$ or $x = a/b$.

$$\text{Domain } (f) = \mathbb{R} - (a/b)$$

$$\text{Let } f(x) = y$$

$$(ax+b)/(bx-a) = y$$

$$ax + b = y(bx - a)$$

$$ax + b = bxy - ay$$

$$ax - bxy = -ay - b$$

$$x(a - by) = -(ay + b)$$

$$\therefore x = -(ay+b)/(a-by)$$

$$\text{When } a - by = 0 \text{ or } y = a/b$$

Hence, $f(x)$ cannot take the value a/b .

$$\therefore \text{Range } (f) = \mathbb{R} - (a/b)$$

$$\text{(ii) } f(x) = (ax-b)/(cx-d)$$

$f(x)$ is defined for all real values of x , except for the case when $cx - d = 0$ or $x = d/c$.

$$\text{Domain } (f) = \mathbb{R} - (d/c)$$

$$\text{Let } f(x) = y$$

$$(ax-b)/(cx-d) = y$$

$$ax - b = y(cx - d)$$

$$ax - b = cxy - dy$$

$$ax - cxy = b - dy$$

$$x(a - cy) = b - dy$$

$$\therefore x = (b-dy)/(a-cy)$$

$$\text{When } a - cy = 0 \text{ or } y = a/c,$$

Hence, $f(x)$ cannot take the value a/c .

$$\therefore \text{Range } (f) = \mathbb{R} - (a/c)$$

$$\text{(iii) } f(x) = \sqrt{x-1}$$

We know the square of a real number is never negative.

$f(x)$ takes real values only when $x - 1 \geq 0$

$$x \geq 1$$

$$\therefore x \in [1, \infty)$$

$$\text{Thus, domain } (f) = [1, \infty)$$

When $x \geq 1$, we have $x - 1 \geq 0$

Hence, $\sqrt{x-1} \geq 0 \Rightarrow f(x) \geq 0$

$$f(x) \in [0, \infty)$$

$$\therefore \text{Range } (f) = [0, \infty)$$

$$\text{(iv) } f(x) = \sqrt{x-3}$$

We know the square of a real number is never negative.

$f(x)$ takes real values only when $x - 3 \geq 0$

$$x \geq 3$$

$$\therefore x \in [3, \infty)$$

$$\text{Domain (f)} = [3, \infty)$$

When $x \geq 3$, we have $x - 3 \geq 0$

Hence, $\sqrt{x-3} \geq 0 \Rightarrow f(x) \geq 0$

$$f(x) \in [0, \infty)$$

$$\therefore \text{Range (f)} = [0, \infty)$$

$$\text{(v)} f(x) = \frac{x-2}{2-x}$$

$f(x)$ is defined for all real values of x , except for the case when $2 - x = 0$ or $x = 2$.

$$\text{Domain (f)} = \mathbb{R} - \{2\}$$

We have, $f(x) = \frac{x-2}{2-x}$

$$\begin{aligned} f(x) &= \frac{-(2-x)}{2-x} \\ &= -1 \end{aligned}$$

When $x \neq 2$, $f(x) = -1$

$$\therefore \text{Range (f)} = \{-1\}$$

$$\text{(vi)} f(x) = |x-1|$$

$$\text{we know } |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Now we have,

$$|x-1| = \begin{cases} -(x-1), & x-1 < 0 \\ x-1, & x-1 \geq 0 \end{cases}$$

$$\therefore f(x) = |x-1| = \begin{cases} 1-x, & x < 1 \\ x-1, & x \geq 1 \end{cases}$$

Hence, $f(x)$ is defined for all real numbers x .

$$\text{Domain (f)} = \mathbb{R}$$

When, $x < 1$, we have $x - 1 < 0$ or $1 - x > 0$.

$$|x-1| > 0 \Rightarrow f(x) > 0$$

When, $x \geq 1$, we have $x - 1 \geq 0$.

$$|x-1| \geq 0 \Rightarrow f(x) \geq 0$$

$$\therefore f(x) \geq 0 \text{ or } f(x) \in [0, \infty)$$

$$\text{Range (f)} = [0, \infty)$$

$$\text{(vii)} f(x) = -|x|$$

$$\text{we know } |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Now we have,

$$-|x| = \begin{cases} -(-x), & x < 0 \\ -x, & x \geq 0 \end{cases}$$

$$\therefore f(x) = -|x| = \begin{cases} x, & x < 0 \\ -x, & x \geq 0 \end{cases}$$

Hence, $f(x)$ is defined for all real numbers x .

Domain (f) = \mathbb{R}

When, $x < 0$, we have $-|x| < 0$

$$f(x) < 0$$

When, $x \geq 0$, we have $-x \leq 0$.

$$|x| \leq 0 \Rightarrow f(x) \leq 0$$

$$\therefore f(x) \leq 0 \text{ or } f(x) \in (-\infty, 0]$$

$$\text{Range } (f) = (-\infty, 0]$$

(viii) $f(x) = \sqrt{9-x^2}$

We know the square of a real number is never negative.

$f(x)$ takes real values only when $9 - x^2 \geq 0$

$$9 \geq x^2$$

$$x^2 \leq 9$$

$$x^2 - 9 \leq 0$$

$$x^2 - 3^2 \leq 0$$

$$(x + 3)(x - 3) \leq 0$$

$$x \geq -3 \text{ and } x \leq 3$$

$$\therefore x \in [-3, 3]$$

$$\text{Domain } (f) = [-3, 3]$$

When, $x \in [-3, 3]$, we have $0 \leq 9 - x^2 \leq 9$

$$0 \leq \sqrt{9-x^2} \leq 3 \Rightarrow 0 \leq f(x) \leq 3$$

$$\therefore f(x) \in [0, 3]$$

$$\text{Range } (f) = [0, 3]$$