Exercise 21(B)

Prove that:

(i) $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\sin A+\cos A$

(ii) $\dfrac{\cos^3 A+\sin^3 A}{\cos A+\sin A}+\dfrac{\cos^3 A-\sin^3 A}{\cos A-\sin A}=2$

(iii) $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}=\sec A\operatorname{cosec} A+1$

(iv) $\left(\tan A+\dfrac{1}{\cos A}\right)^2+\left(\tan A-\dfrac{1}{\cos A}\right)^2=2\left(\dfrac{1+\sin^2 A}{1-\sin^2 A}\right)$

(v) $2\sin^2 A+\cos^4 A=1+\sin^4 A$

(vi) $\dfrac{\sin A-\sin B}{\cos A+\cos B}+\dfrac{\cos A-\cos B}{\sin A+\sin B}=0$

(vii) $(\operatorname{cosec} A-\sin A)(\sec A-\cos A)=\dfrac{1}{\tan A+\cot A}$

(viii) $(1+\tan A\tan B)^2+(\tan A-\tan B)^2=\sec^2 A\sec^2 B$

(ix) $\dfrac{1}{\cos A+\sin A-1}+\dfrac{1}{\cos A+\sin A+1}=\operatorname{cosec} A+\sec A$

Solution:

(i)

$$\text{LHS}=\frac{\cos A}{1-\tan A}+\frac{\sin A}{1-\cot A}$$

$$=\frac{\cos A}{1-\frac{\sin A}{\cos A}}+\frac{\sin A}{1-\frac{\cos A}{\sin A}}=\frac{\cos A}{\frac{\cos A-\sin A}{\cos A}}+\frac{\sin A}{\frac{\sin A-\cos A}{\sin A}}$$

$$=\frac{\cos^2 A}{\cos A-\sin A}+\frac{\sin^2 A}{\sin A-\cos A}=\frac{\cos^2 A-\sin^2 A}{(\cos A-\sin A)}$$

$$=\sin A+\cos A=\text{RHS}$$

\- Hence Proved

(ii) Taking LHS,

$$\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A} + \frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}$$

$$= \frac{(\cos^3 A + \sin^3 A)(\cos A - \sin A) + (\cos^3 A - \sin^3 A)(\cos A + \sin A)}{\cos^2 A - \sin^2 A}$$

$$= \frac{\cos^4 A - \cos^3 A \sin A + \sin^3 A \cos A - \sin^4 A + \cos^4 A + \cos^3 A \sin A - \sin^3 A \cos A - \sin^4 A}{\cos^2 A - \sin^2 A}$$

$$= \frac{2(\cos^4 A - \sin^4 A)}{\cos^2 A - \sin^2 A}$$

$$= \frac{2(\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)}{(\cos^2 A - \sin^2 A)}$$

$$= 2(\cos^2 A + \sin^2 A)$$

$$= 2 \ (\because \cos^2 A + \sin^2 A = 1)$$

\- Hence Proved

(iii) $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$

$$= \frac{\tan A}{1 - \frac{1}{\tan A}} + \frac{\frac{1}{\tan A}}{1 - \tan A}$$

$$= \frac{\tan^2 A}{\tan A - 1} + \frac{1}{\tan A(1 - \tan A)}$$

$$= \frac{\tan^3 A - 1}{\tan A(\tan A - 1)}$$

$$= \frac{(\tan A - 1)(\tan^2 A + 1 + \tan A)}{\tan A(\tan A - 1)}$$

$$= \frac{\sec^2 A + \tan A}{\tan A}$$

$$= \frac{\frac{1}{\cos^2 A}}{\frac{\sin A}{\cos A}} + 1$$

$$= \frac{1}{\sin A \cos A} + 1$$

$$= \sec A \operatorname{cosec} A + 1$$

\- Hence Proved

(iv) $\left(\tan A + \frac{1}{\cos A}\right)^2 + \left(\tan A - \frac{1}{\cos A}\right)^2$

$= \left(\frac{\sin A + 1}{\cos A}\right)^2 + \left(\frac{\sin A - 1}{\cos A}\right)^2$

$= \frac{\sin^2 A + 1 + 2\sin A + \sin^2 A + 1 - 2\sin A}{\cos^2 A}$

$= \frac{2 + 2\sin^2 A}{\cos^2 A}$

$= 2\left(\frac{1 + \sin^2 A}{1 - \sin^2 A}\right)$

\- Hence Proved

(v) Taking LHS,

$2\sin^2 A + \cos^2 A$

$= 2\sin^2 A + (1 - \sin^2 A)^2$

$= 2\sin^2 A + 1 + \sin^4 A - 2\sin^2 A$

$= 1 + \sin^4 A$ = RHS

\- Hence Proved

(vi) $\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$

$= \frac{\sin^2 A - \sin^2 B + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$

$= \frac{(\sin^2 A + \cos^2 A) - (\sin^2 B + \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)}$

$= \frac{1 - 1}{(\cos A + \cos B)(\sin A + \sin B)}$

$= 0$

\- Hence Proved

(vii) LHS

$= (\operatorname{cosec} A - \sin A)(\sec A - \cos A)$

$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)$

$= \left(\frac{1 - \sin^2 A}{\sin A}\right)\left(\frac{1 - \cos^2 A}{\cos A}\right)$

$= \left(\frac{\cos^2 A}{\sin A}\right)\left(\frac{\sin^2 A}{\cos A}\right)$

$= \sin A \cos A$

$$\text{RHS} = \frac{1}{\tan A + \cot A}$$

$$= \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}}$$

$$= \frac{\sin A \cos A}{\sin^2 A + \cos^2 A}$$

$$= \sin A \cos A$$

LHS = RHS

\- Hence Proved

(viii) $(1 + \tan A \tan B)^2 + (\tan A - \tan B)^2$

$= 1 + \tan^2 A \tan^2 B + 2 \tan A \tan B + \tan^2 A + \tan^2 B - 2 \tan A \tan B$

$= 1 + \tan^2 A + \tan^2 B + \tan^2 A \tan^2 B$

$= \sec^2 A + \tan^2 B(1 + \tan^2 A)$

$= \sec^2 A + \tan^2 B \sec^2 A$

$= \sec^2 A(1 + \tan^2 B)$

$= \sec^2 A \sec^2 B$

\- Hence Proved

(ix) $$\frac{1}{(\cos A + \sin A) - 1} + \frac{1}{(\cos A + \sin A) + 1}$$

$$= \frac{\cos A + \sin A + 1 + \cos A + \sin A - 1}{(\cos A + \sin A)^2 - 1}$$

$$= \frac{2(\cos A + \sin A)}{\cos^2 A + \sin^2 A + 2 \cos A \sin A - 1}$$

$$= \frac{2(\cos A + \sin A)}{1 + 2 \cos A \sin A - 1} = \frac{\cos A + \sin A}{\cos A \sin A}$$

$$= \frac{\cos A}{\cos A \sin A} + \frac{\sin A}{\cos A \sin A}$$

$$= \frac{1}{\sin A} + \frac{1}{\cos A}$$

$$= \operatorname{cosec} A + \sec A$$

\- Hence Proved

1. If x cos A + y sin A = m and x sin A – y cos A = n, then prove that:
$x^2 + y^2 = m^2 + n^2$
Solution:

Taking RHS,
$m^2 + n^2$
$= (x \cos A + y \sin A)^2 + (x \sin A - y \cos A)^2$

$= x^2 \cos^2 A + y^2 \sin^2 A + 2xy \cos A \sin A + x^2 \sin^2 A + y^2 \cos^2 A – 2xy \sin A \cos A$
$= x^2 (\cos^2 A + \sin^2 A) + y^2 (\sin^2 A + \cos^2 A)$
$= x^2 + y^2$ [Since, $\cos^2 A + \sin^2 A = 1$]
= RHS

2. If m = a sec A + b tan A and n = a tan A + b sec A, prove that $m^2 – n^2 = a^2 – b^2$
Solution:

Taking LHS,
$m^2 – n^2$
$= (a \sec A + b \tan A)^2 – (a \tan A + b \sec A)^2$
$= a^2 \sec^2 A + b^2 \tan^2 A + 2 ab \sec A \tan A – a^2 \tan^2 A – b^2 \sec^2 A – 2ab \tan A \sec A$
$= a^2 (\sec^2 A - \tan^2 A) + b^2 (\tan^2 A - \sec^2 A)$
$= a^2 (1) + b^2 (-1)$ [Since, $\sec^2 A - \tan^2 A = 1$]
$= a^2 – b^2$
= RHS