EXERCISE 24.2

1. Find the coordinates of the centre radius of each of the following circle:

(i) $x^2 + y^2 + 6x - 8y - 24 = 0$

(ii) $2x^2 + 2y^2 – 3x + 5y = 7$

$$(iii)\frac{1}{2}(x^2 + y^2) + x\ cos\theta + y\ sin\theta - 4 = 0$$

(iv) $x^2 + y^2 – ax – by = 0$

Solution:

(i) $x^2 + y^2 + 6x - 8y - 24 = 0$

Given:

The equation of the circle is $x^2 + y^2 + 6x - 8y - 24 = 0$ …… (1)

We know that for a circle $x^2 + y^2 + 2ax + 2by + c = 0$ …… (2)

Centre = (-a, -b)

So by comparing equation (1) and (2)

Centre = (-6/2, -(-8)/2)

= (-3, 4)

Radius = $\sqrt{(a^2 + b^2 - c)}$

= $\sqrt{(3^2 + 4^2 – (-24))}$

= $\sqrt{(9 + 16 + 24)}$

= $\sqrt{(49)}$

= 7

∴ The centre of the circle is (-3, 4) and the radius is 7.

(ii) $2x^2 + 2y^2 – 3x + 5y = 7$

Given:

The equation of the circle is $2x^2 + 2y^2 – 3x + 5y = 7$ (divide by 2 we get)

$x^2 + y^2 – 3x/2 + 5y/2 = 7/2$

Now, by comparing with the equation $x^2 + y^2 + 2ax + 2by + c = 0$

Centre = (-a, -b)

Centre = $\left(\frac{-\left(\frac{-3}{2}\right)}{2}, \frac{-\left(\frac{5}{2}\right)}{2}\right)$

= $\left(\frac{3}{4}, \frac{-5}{4}\right)$

Radius = $\sqrt{(a^2 + b^2 - c)}$

Radius = $\sqrt{\left(\frac{3}{4}\right)^2 + \left(-\frac{5}{4}\right)^2 - \left(\frac{-7}{2}\right)}$

= $\sqrt{\frac{9}{16} + \frac{25}{16} + \frac{7}{2}}$

= $\sqrt{\frac{90}{16}}$

= $\frac{3\sqrt{10}}{4}$

∴ The centre and radius of the circle is $\left(\frac{3}{4}, \frac{-5}{4}\right)$ and $\frac{3\sqrt{10}}{4}$.

(iii) $\boldsymbol{\frac{1}{2}(x^2 + y^2) + x\,cos\theta + y\,sin\theta - 4 = 0}$

Given:

The equation of the circle is

$\frac{1}{2}(x^2 + y^2) + x\,cos\theta + y\,sin\theta - 4 = 0$

(Multiply by 2 we get)

$x^2 + y^2 + 2x \cos \theta + 2y \sin \theta - 8 = 0$

By comparing with the equation $x^2 + y^2 + 2ax + 2by + c = 0$

Centre = (-a, -b)

= $[(-2\cos \theta)/2, (-2\sin \theta)/2]$

= $(-\cos \theta, -\sin \theta)$

Radius = $\sqrt{(a^2 + b^2 - c)}$

= $\sqrt{[(-\cos \theta)^2 + (\sin \theta)^2 - (-8)]}$

= $\sqrt{[\cos^2 \theta + \sin^2 \theta + 8]}$

= $\sqrt{[1 + 8]}$

= $\sqrt{[9]}$

= 3

∴ The centre and radius of the circle is $(-\cos \theta, -\sin \theta)$ and 3.

(iv) $x^2 + y^2 - ax - by = 0$

Given:
Equation of the circle is $x^2 + y^2 - ax - by = 0$
By comparing with the equation $x^2 + y^2 + 2ax + 2by + c = 0$
Centre = (-a, -b)
= (-(-a)/2, -(-b)/2)
= (a/2, b/2)

Radius = $\sqrt{(a^2 + b^2 - c)}$
= $\sqrt{[(a/2)^2 + (b/2)^2]}$
= $\sqrt{[(a^2/4 + b^2/4)]}$
= $\sqrt{[(a^2 + b^2)/4]}$
= $[\sqrt{(a^2 + b^2)}]/2$
∴ The centre and radius of the circle is (a/2, b/2) and $[\sqrt{(a^2 + b^2)}]/2$

2. Find the equation of the circle passing through the points :
(i) (5, 7), (8, 1) and (1, 3)
(ii) (1, 2), (3, - 4) and (5, - 6)
(iii) (5, -8), (-2, 9) and (2, 1)
(iv) (0, 0), (-2, 1) and (-3, 2)
Solution:
(i) (5, 7), (8, 1) and (1, 3)
By using the standard form of the equation of the circle:
$x^2 + y^2 + 2ax + 2by + c = 0$ ......(1)
Firstly let us find the values of a, b and c
Substitute the given point (5, 7) in equation (1), we get
$5^2 + 7^2 + 2a(5) + 2b(7) + c = 0$
25 + 49 + 10a + 14b + c = 0
10a + 14b + c + 74 = 0 .....(2)

By substituting the given point (8, 1) in equation (1), we get
$8^2 + 1^2 + 2a(8) + 2b(1) + c = 0$
64 + 1 + 16a + 2b + c = 0
16a + 2b + c + 65 = 0......(3)

Substituting the point (1, 3) in equation (1), we get
$1^2 + 3^2 + 2a(1) + 2b(3) + c = 0$
1 + 9 + 2a + 6b + c = 0
2a + 6b + c + 10 = 0. ....(4)

Now by simplifying the equations (2), (3), (4) we get the values
a = -29/6, b = -19/6, c = 56/3

Substituting the values of a, b, c in equation (1), we get
$x^2 + y^2 + 2(-29/6)x + 2(-19/6) + 56/3 = 0$
$x^2 + y^2 – 29x/3 – 19y/3 + 56/3 = 0$
$3x^2 + 3y^2 - 29x - 19y + 56 = 0$
∴ The equation of the circle is $3x^2 + 3y^2 - 29x - 19y + 56 = 0$

(ii) (1, 2), (3, - 4) and (5, - 6)
By using the standard form of the equation of the circle:
$x^2 + y^2 + 2ax + 2by + c = 0$ ......(1)
Substitute the points (1, 2) in equation (1), we get
$1^2 + 2^2 + 2a(1) + 2b(2) + c = 0$
$1 + 4 + 2a + 4b + c = 0$
$2a + 4b + c + 5 = 0$......(2)

Substitute the points (3, -4) in equation (1), we get
$3^2 + (-4)^2 + 2a(3) + 2b(-4) + c = 0$
$9 + 16 + 6a - 8b + c = 0$
$6a - 8b + c + 25 = 0$. ....(3)

Substitute the points (5, -6) in equation (1), we get
$5^2 + (-6)^2 + 2a(5) + 2b(-6) + c = 0$
$25 + 36 + 10a - 12b + c = 0$
$10a - 12b + c + 61 = 0$ .....(4)

Now by simplifying the equations (2), (3), (4) we get
a = - 11, b = - 2, c = 25

Substitute the values of a, b and c in equation (1), we get
$x^2 + y^2 + 2(-11)x + 2(-2) + 25 = 0$
$x^2 + y^2 - 22x - 4y + 25 = 0$
∴ The equation of the circle is $x^2 + y^2 - 22x - 4y + 25 = 0$

(iii) (5, -8), (-2, 9) and (2, 1)
By using the standard form of the equation of the circle:
$x^2 + y^2 + 2ax + 2by + c = 0$ ......(1)
Substitute the point (5, -8) in equation (1), we get

$5^2 + (-8)^2 + 2a(5) + 2b(-8) + c = 0$
$25 + 64 + 10a - 16b + c = 0$
$10a - 16b + c + 89 = 0$ .....(2)

Substitute the points (-2, 9) in equation (1), we get
$(-2)^2 + 9^2 + 2a(-2) + 2b(9) + c = 0$
$4 + 81 - 4a + 18b + c = 0$
$-4a + 18b + c + 85 = 0$. .... (3)

Substitute the points (2, 1) in equation (1), we get
$2^2 + 1^2 + 2a(2) + 2b(1) + c = 0$
$4 + 1 + 4a + 2b + c = 0$
$4a + 2b + c + 5 = 0$......(4)

By simplifying equations (2), (3), (4) we get
a = 58, b = 24, c = - 285.

Now, by substituting the values of a, b, c in equation (1), we get
$x^2 + y^2 + 2(58)x + 2(24) - 285 = 0$
$x^2 + y^2 + 116x + 48y - 285 = 0$
∴ The equation of the circle is $x^2 + y^2 + 116x + 48y - 285 = 0$

(iv) (0, 0), (-2, 1) and (-3, 2)
By using the standard form of the equation of the circle:
$x^2 + y^2 + 2ax + 2by + c = 0$ ......(1)
Substitute the points (0, 0) in equation (1), we get
$0^2 + 0^2 + 2a(0) + 2b(0) + c = 0$
$0 + 0 + 0a + 0b + c = 0$
$c = 0$. .... (2)

Substitute the points (-2, 1) in equation (1), we get
$(-2)^2 + 1^2 + 2a(-2) + 2b(1) + c = 0$
$4 + 1 - 4a + 2b + c = 0$
$-4a + 2b + c + 5 = 0$. .... (3)

Substitute the points (-3, 2) in equation (1), we get
$(-3)^2 + 2^2 + 2a(-3) + 2b(2) + c = 0$
$9 + 4 - 6a + 4b + c = 0$
$-6a + 4b + c + 13 = 0$. .... (4)

By simplifying the equations (2), (3), (4) we get
a = -3/2, b = -11/2, c = 0

Now, by substituting the values of a, b, c in equation (1), we get
$x^2 + y^2 + 2(-3/2)x + 2(-11/2)y + 0 = 0$
$x^2 + y^2 - 3x - 11y = 0$
∴ The equation of the circle is $x^2 + y^2 - 3x - 11y = 0$

3. Find the equation of the circle which passes through (3, - 2), (- 2, 0) and has its centre on the line 2x – y = 3.
Solution:
Given:
The line 2x – y = 3 … (1)
The points (3, -2), (-2, 0)
By using the standard form of the equation of the circle:
$x^2 + y^2 + 2ax + 2by + c = 0$ ......(2)
Let us substitute the centre (-a, -b) in equation (1) we get,
2(- a) - (- b) = 3
-2a + b = 3
2a - b + 3 = 0. .....(3)

Now Substitute the given points (3, -2) in equation (2), we get
$3^2 + (- 2)^2 + 2a(3) + 2b(- 2) + c = 0$
9 + 4 + 6a - 4b + c = 0
6a - 4b + c + 13 = 0. ....(4)

Substitute the points (-2, 0) in equation (2), we get
$(- 2)^2 + 0^2 + 2a(- 2) + 2b(0) + c = 0$
4 + 0 - 4a + c = 0
4a - c - 4 = 0......(5)

By simplifying the equations (3), (4) and (5) we get,
a = 3/2, b = 6, c = 2

Again by substituting the values of a, b, c in (2), we get
$x^2 + y^2 + 2 (3/2)x + 2 (6)y + 2 = 0$
$x^2 + y^2 + 3x + 12y + 2 = 0$
∴ The equation of the circle is $x^2 + y^2 + 3x + 12y + 2 = 0$.

4. Find the equation of the circle which passes through the points (3, 7), (5, 5) and has its centre on line x – 4y = 1.

Solution:

Given:

The points (3, 7), (5, 5)

The line x – 4y = 1…. (1)

By using the standard form of the equation of the circle:

$x^2 + y^2 + 2ax + 2by + c = 0$ ......(2)

Let us substitute the centre (-a, -b) in equation (1) we get,

(- a) - 4(- b) = 1

-a + 4b = 1

a - 4b + 1 = 0 ......(3)

Substitute the points (3, 7) in equation (2), we get

$3^2 + 7^2 + 2a(3) + 2b(7) + c = 0$

9 + 49 + 6a + 14b + c = 0

6a + 14b + c + 58 = 0......(4)

Substitute the points (5, 5) in equation (2), we get

$5^2 + 5^2 + 2a(5) + 2b(5) + c = 0$

25 + 25 + 10a + 10b + c = 0

10a + 10b + c + 50 = 0......(5)

By simplifying equations (3), (4) and (5) we get,

a = 3, b = 1, c = - 90

Now, by substituting the values of a, b, c in equation (2), we get

$x^2 + y^2 + 2(3)x + 2(1)y - 90 = 0$

$x^2 + y^2 + 6x + 2y - 90 = 0$

∴ The equation of the circle is $x^2 + y^2 + 6x + 2y - 90 = 0$.

5. Show that the points (3, -2), (1, 0), (-1, -2) and (1, -4) are con - cyclic.

Solution:

Given:

The points (3, -2), (1, 0), (-1, -2) and (1, -4)

Let us assume the circle passes through the points A, B, C.

So by using the standard form of the equation of the circle:

$x^2 + y^2 + 2ax + 2by + c = 0$ (1)

Substitute the points A (3, - 2) in equation (1), we get,

$3^2 + (-2)^2 + 2a(3) + 2b(-2) + c = 0$
$9 + 4 + 6a - 4b + c = 0$
$6a - 4b + c + 13 = 0$. ....(2)

Substitute the points B (1, 0) in equation (1), we get,
$1^2 + 0^2 + 2a(1) + 2b(0) + c = 0$
$1 + 2a + c = 0$.... ..- (3)

Substitute the points C (-1, -2) in equation (1), we get,
$(- 1)^2 + (- 2)^2 + 2a(- 1) + 2b(- 2) + c = 0$
$1 + 4 - 2a - 4b + c = 0$
$5 - 2a - 4b + c = 0$
$2a + 4b - c - 5 = 0$ .....(4)

Upon simplifying the equations (2), (3) and (4) we get,
$a = - 1$, $b = 2$ and $c = 1$

Substituting the values of a, b, c in equation (1), we get
$x^2 + y^2 + 2(- 1)x + 2(2)y + 1 = 0$
$x^2 + y^2 - 2x + 4y + 1 = 0$ ...... (5)

Now by substituting the point D (1, -4) in equation (5) we get,
$1^2 + (- 4)^2 - 2(1) + 4(- 4) + 1$
$1 + 16 - 2 - 16 + 1$
0
∴ The points (3, -2), (1, 0), (-1, -2), (1, -4) are con - cyclic.

6. Show that the points (5, 5), (6, 4), (- 2, 4) and (7, 1) all lie on a circle, and find its equation, centre, and radius.
Solution:
Given:
The points (5, 5), (6, 4), (- 2, 4) and (7, 1) all lie on a circle.
Let us assume the circle passes through the points A, B, C.
So by using the standard form of the equation of the circle:
$x^2 + y^2 + 2ax + 2by + c = 0$ .....(1)

Substituting A (5, 5) in (1), we get,
$5^2 + 5^2 + 2a(5) + 2b(5) + c = 0$
$25 + 25 + 10a + 10b + c = 0$

$10a + 10b + c + 50 = 0$. .... (2)

Substitute the points B (6, 4) in equation (1), we get,
$6^2 + 4^2 + 2a(6) + 2b(4) + c = 0$
$36 + 16 + 12a + 8b + c = 0$
$12a + 8b + c + 52 = 0$. .... (3)

Substitute the point C (-2, 4) in equation (1), we get,
$(-2)^2 + 4^2 + 2a(-2) + 2b(4) + c = 0$
$4 + 16 - 4a + 8b + c = 0$
$20 - 4a + 8b + c = 0$
$4a - 8b - c - 20 = 0$. .... (4)

Upon simplifying equations (2), (3) and (4) we get,
$a = -2$, $b = -1$ and $c = -20$

Now by substituting the values of a, b, c in equation (1), we get
$x^2 + y^2 + 2(-2)x + 2(-1)y - 20 = 0$
$x^2 + y^2 - 4x - 2y - 20 = 0$ ...... (5)

Substituting D (7, 1) in equation (5) we get,
$7^2 + 1^2 - 4(7) - 2(1) - 20$
$49 + 1 - 28 - 2 - 20$
0
∴ The points (3, -2), (1, 0), (-1, -2), (1, -4) lie on a circle.

Now let us find the centre and the radius.
We know that for a circle $x^2 + y^2 + 2ax + 2by + c = 0$,
Centre = (-a, -b)
Radius = $\sqrt{(a^2 + b^2 - c)}$

Comparing equation (5) with equation (1), we get
Centre = [-(-4)/2, -(-2)/2)]
= (2, 1)
Radius = $\sqrt{(2^2 + 1^2 - (-20))}$
= $\sqrt{(25)}$
= 5
∴ The centre and radius of the circle is (2, 1) and 5.

7. Find the equation of the circle which circumscribes the triangle formed by the lines:
(i) x + y + 3 = 0, x - y + 1 = 0 and x = 3
(ii) 2x + y - 3 = 0, x + y - 1 = 0 and 3x + 2y - 5 = 0
(iii) x + y = 2, 3x - 4y = 6 and x - y = 0
(iv) y = x + 2, 3y = 4x and 2y = 3x
Solution:
(i) x + y + 3 = 0, x - y + 1 = 0 and x = 3
Given:
The lines x + y + 3 = 0
x - y + 1 = 0
x = 3

On solving these lines we get the intersection points A (-2, -1), B (3, 4), C (3, -6)
So by using the standard form of the equation of the circle:
$x^2 + y^2 + 2ax + 2by + c = 0$......(1)

Substitute the points (-2, -1) in equation (1), we get
$(- 2)^2 + (- 1)^2 + 2a(-2) + 2b(-1) + c = 0$
$4 + 1 - 4a - 2b + c = 0$
$5 - 4a - 2b + c = 0$
$4a + 2b - c - 5 = 0$ .....(2)

Substitute the points (3, 4) in equation (1), we get
$3^2 + 4^2 + 2a(3) + 2b(4) + c = 0$
$9 + 16 + 6a + 8b + c = 0$
$6a + 8b + c + 25 = 0$......(3)

Substitute the points (3, -6) in equation (1), we get
$3^2 + (- 6)^2 + 2a(3) + 2b(- 6) + c = 0$
$9 + 36 + 6a - 12b + c = 0$
$6a - 12b + c + 45 = 0$. ....(4)

Upon simplifying equations (2), (3), (4) we get
a = - 3, b = 1, c = -15.

Now by substituting the values of a, b, c in equation (1), we get
$x^2 + y^2 + 2(- 3)x + 2(1)y - 15 = 0$
$x^2 + y^2 - 6x + 2y - 15 = 0$

∴ The equation of the circle is $x^2 + y^2 - 6x + 2y - 15 = 0$.

(ii) 2x + y - 3 = 0, x + y - 1 = 0 and 3x + 2y - 5 = 0

Given:

The lines $2x + y - 3 = 0$

$x + y - 1 = 0$

$3x + 2y - 5 = 0$

On solving these lines we get the intersection points A(2, - 1), B(3, - 2), C(1,1)

So by using the standard form of the equation of the circle:

$x^2 + y^2 + 2ax + 2by + c = 0$......(1)

Substitute the points (2, -1) in equation (1), we get

$2^2 + (- 1)^2 + 2a(2) + 2b(- 1) + c = 0$

$4 + 1 + 4a - 2b + c = 0$

$4a - 2b + c + 5 = 0$. ..... (2)

Substitute the points (3, -2) in equation (1), we get

$3^2 + (- 2)^2 + 2a(3) + 2b(- 2) + c = 0$

$9 + 4 + 6a - 4b + c = 0$

$6a - 4b + c + 13 = 0$. ..... (3)

Substitute the points (1, 1) in equation (1), we get

$1^2 + 1^2 + 2a(1) + 2b(1) + c = 0$

$1 + 1 + 2a + 2b + c = 0$

$2a + 2b + c + 2 = 0$ ..... (4)

Upon simplifying equations (2), (3), (4) we get

$a = -13/2$, $b = -5/2$, $c = 16$

Now by substituting the values of a, b, c in equation (1), we get

$x^2 + y^2 + 2(-13/2)x + 2(-5/2)y + 16 = 0$

$x^2 + y^2 - 13x - 5y + 16 = 0$

∴ The equation of the circle is $x^2 + y^2 - 13x - 5y + 16 = 0$

(iii) x + y = 2, 3x - 4y = 6 and x - y = 0

Given:

The lines $x + y = 2$

$3x - 4y = 6$

$x - y = 0$

On solving these lines we get the intersection points A(2,0), B(- 6, - 6), C(1,1)
So by using the standard form of the equation of the circle:
$x^2 + y^2 + 2ax + 2by + c = 0$......(1)

Substitute the points (2, 0) in equation (1), we get
$2^2 + 0^2 + 2a(2) + 2b(0) + c = 0$
$4 + 4a + c = 0$
$4a + c + 4 = 0$......(2)

Substitute the point (-6, -6) in equation (1), we get
$(- 6)^2 + (- 6)^2 + 2a(- 6) + 2b(- 6) + c = 0$
$36 + 36 - 12a - 12b + c = 0$
$12a + 12b - c - 72 = 0$. ....(3)

Substitute the points (1, 1) in equation (1), we get
$1^2 + 1^2 + 2a(1) + 2b(1) + c = 0$
$1 + 1 + 2a + 2b + c = 0$
$2a + 2b + c + 2 = 0$......(4)

Upon simplifying equations (2), (3), (4) we get
$a = 2, b = 3, c = - 12$.

Substituting the values of a, b, c in equation (1), we get
$x^2 + y^2 + 2(2)x + 2(3)y - 12 = 0$
$x^2 + y^2 + 4x + 6y - 12 = 0$
∴ The equation of the circle is $x^2 + y^2 + 4x + 6y - 12 = 0$

(iv) $y = x + 2$, $3y = 4x$ and $2y = 3x$
Given:
The lines $y = x + 2$
$3y = 4x$
$2y = 3x$

On solving these lines we get the intersection points A(6,8), B(0,0), C(4,6)
So by using the standard form of the equation of the circle:
$x^2 + y^2 + 2ax + 2by + c = 0$......(1)

Substitute the points (6, 8) in equation (1), we get
$6^2 + 8^2 + 2a(6) + 2b(8) + c = 0$
$36 + 64 + 12a + 16b + c = 0$
$12a + 16b + c + 100 = 0$.......(2)

Substitute the points (0, 0) in equation (1), we get
$0^2 + 0^2 + 2a(0) + 2b(0) + c = 0$
$0 + 0 + 0a + 0b + c = 0$
$c = 0$..... (3)

Substitute the points (4, 6) in equation (1), we get
$4^2 + 6^2 + 2a(4) + 2b(6) + c = 0$
$16 + 36 + 8a + 12b + c = 0$
$8a + 12b + c + 52 = 0$. ....(4)

Upon simplifying equations (2), (3), (4) we get
$a = -23$, $b = 11$, $c = 0$

Now by substituting the values of a, b, c in equation (1), we get
$x^2 + y^2 + 2(-23)x + 2(11)y + 0 = 0$
$x^2 + y^2 - 46x + 22y = 0$
∴ The equation of the circle is $x^2 + y^2 - 46x + 22y = 0$