

**EXERCISE 7.1**

Find the adjoint of each of the following matrices:

(i)  $\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$

(ii)  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(iii)  $\begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

(iv)  $\begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix}$

 Verify that  $(\text{adj } A) A = |A| I = A (\text{adj } A)$  for the above matrices.

Solution:

(i) Let

$$A = \begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$$

Cofactors of A are

$$C_{11} = 4$$

$$C_{12} = -2$$

$$C_{21} = -5$$

$$C_{22} = -3$$

$$\text{Since, adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} 4 & -2 \\ -5 & -3 \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A) A = \begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -12 - 10 & 20 - 20 \\ 6 - 6 & -10 - 12 \end{bmatrix}$$

$$(\text{adj } A) A = \begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix}$$

$$\text{And, } |A| I = \begin{vmatrix} -3 & 5 \\ 2 & 4 \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (-22) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix}$$

$$\text{Also, } A(\text{adj } A) = \begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -12 - 10 & 20 - 20 \\ 6 - 6 & -10 - 12 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix}$$

$$\text{Hence, } (\text{adj } A)A = |A|I = A(\text{adj } A)$$

(ii) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Therefore cofactors of A are

$$C_{11} = d$$

$$C_{12} = -c$$

$$C_{21} = -b$$

$$C_{22} = a$$

We know that,  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

Therefore by substituting these values we get,

$$(\text{adj } A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$$

$$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Now, } (\text{adj } A)A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad - bc & bd - bd \\ -ac + ac & -bc + ad \end{bmatrix}$$

$$(\text{adj } A)A = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$\text{And, } |A|I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

Also,

$$A (\text{adj } A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

Hence,  $(\text{adj } A) A = |A| I = A (\text{adj } A)$

(iii) Let

$$A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

Therefore cofactors of A are

$$C_{11} = \cos \alpha$$

$$C_{12} = -\sin \alpha$$

$$C_{21} = -\sin \alpha$$

$$C_{22} = \cos \alpha$$

We know that,  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

$$(\text{adj } A) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}^T \\ = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\text{Now, } (\text{adj } A) A = \begin{vmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & \cos \alpha \end{vmatrix} \begin{vmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{vmatrix}$$

$$= \begin{bmatrix} -\sin^2 \alpha + \cos^2 \alpha & \cos \alpha \sin \alpha - \sin \alpha \cos \alpha \\ -\cos \alpha \sin \alpha + \sin \alpha \cos \alpha & -\sin^2 \alpha + \cos^2 \alpha \end{bmatrix}$$

$$(\text{adj } A) A = \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$$

$$\text{And, } |A| = \begin{vmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (\cos^2 \alpha - \sin^2 \alpha) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \alpha - \sin^2 \alpha & 0 \\ 0 & \cos^2 \alpha - \sin^2 \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$$

Also,  $A(\text{adj } A)$

$$= \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} \cos^2 \alpha - \sin^2 \alpha & 0 \\ 0 & \cos^2 \alpha - \sin^2 \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$$

Hence,  $(\text{adj } A)A = |A|I = A(\text{adj } A)$

(v) Let

$$A = \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

Therefore cofactors of  $A$  are

$$C_{11} = 1$$

$$C_{12} = \tan \frac{\alpha}{2}$$

$$C_{21} = -\tan \frac{\alpha}{2}$$

$$C_{22} = 1$$

We know that,  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

$$(\text{adj } A) = \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A)A = \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \tan^2 \frac{\alpha}{2} & \tan \frac{\alpha}{2} & \tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & \tan \frac{\alpha}{2} & 1 + \tan^2 \frac{\alpha}{2} \end{bmatrix}$$

$$(\text{adj } A)A = \begin{bmatrix} 1 + \tan^2 \frac{\alpha}{2} & 0 \\ 0 & 1 + \tan^2 \frac{\alpha}{2} \end{bmatrix}$$

$$\text{And, } |A|^{-1} = \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (1 + \tan^2 \frac{\alpha}{2}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \tan^2 \frac{\alpha}{2} & 0 \\ 0 & 1 + \tan^2 \frac{\alpha}{2} \end{bmatrix}$$

$$\text{Also, } A(\text{adj } A) = \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \tan^2 \frac{\alpha}{2} & \tan \frac{\alpha}{2} - \tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} - \tan \frac{\alpha}{2} & 1 + \tan^2 \frac{\alpha}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \tan^2 \frac{\alpha}{2} & 0 \\ 0 & 1 + \tan^2 \frac{\alpha}{2} \end{bmatrix}$$

Hence,  $(\text{adj } A)A = |A|^{-1} = A(\text{adj } A)$

2. Compute the adjoint of each of the following matrices.

$$(i) \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

Solution:

(i) Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Therefore cofactors of A are

$$C_{11} = -3$$

$$C_{21} = 2$$

$$C_{31} = 2$$

$$C_{12} = 2$$

$$C_{22} = -3$$

$$C_{32} = 2$$

$$C_{13} = 2$$

$$C_{23} = 2$$

$$C_{33} = -3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A) A = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 + 4 + 4 & -6 + 2 + 4 & -6 + 4 + 2 \\ 2 - 3 + 4 & 4 - 3 + 4 & 4 - 6 + 2 \\ 2 + 4 - 6 & 4 + 2 - 6 & 4 + 4 - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\text{Also, } |A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-3 + 4 + 4) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\text{Then, } A(\text{adj } A) = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 + 4 + 4 & -6 + 2 + 4 & -6 + 4 + 2 \\ 2 - 3 + 4 & 4 - 3 + 4 & 4 - 6 + 2 \\ 2 + 4 - 6 & 4 + 2 - 6 & 4 + 4 - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\text{Since, } (\text{adj } A) A = |A| I = A (\text{adj } A)$$

(ii) Let

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Cofactors of A

$$C_{11} = 2$$

$$C_{12} = 3$$

$$C_{13} = -13$$

$$C_{21} = -3$$

$$C_{22} = 6$$

$$C_{23} = 9$$

$$C_{31} = 5$$

$$C_{32} = -3$$

$$C_{33} = -1$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & 1 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A) A = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+6+13 & 4+9-13 & 10+3-13 \\ -3+12-9 & -6+18+9 & -15+6+9 \\ 5-3+1 & 10-9+1 & 25-3-1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$$

$$\text{Also, } |A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= [1(3-1) - 2(2+1) + 5(2+3)] \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 21 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$$

$$\begin{aligned} \text{Then, } A(\text{adj } A) &= \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2-6+25 & 3+12-15 & -13+18-5 \\ 4-9+5 & 6+18-3 & -26+27-1 \\ -2-3+5 & -3+6-3 & 13+9-1 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix} \end{aligned}$$

Hence,  $(\text{adj } A)A = |A|I = A(\text{adj } A)$

(iii) Let

$$A = \begin{bmatrix} 4 & -2 & 0 \\ 0 & 4 & -1 \end{bmatrix}$$

Therefore cofactors of A

$$C_{11} = -22$$

$$C_{21} = 11$$

$$C_{31} = -11$$

$$C_{12} = 4$$

$$C_{22} = -2$$

$$C_{32} = 2$$

$$C_{13} = 16$$

$$C_{23} = -8$$

$$C_{33} = 8$$

$$\text{We know that } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Now by substituting the values in above matrix we get,

$$= \begin{bmatrix} -22 & 4 & 16 \\ 11 & -2 & -8 \\ -11 & 2 & 8 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A) A = \begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -44 + 44 + 0 & 22 + 22 - 44 & -66 + 55 + 11 \\ 8 - 8 + 0 & -4 - 4 + 8 & 12 - 10 - 2 \\ 32 - 32 + 0 & -16 - 16 + 32 & 48 - 40 - 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{vmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= -2(-2 - 20) + 1(-4 - 0) + 3[16 - 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (-44 - 4 + 48) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Then, } A (\text{adj } A) = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -44 - 4 + 48 & 22 + 2 - 24 & -22 - 2 + 24 \\ -88 + 8 + 80 & 44 - 4 - 40 & -44 + 4 + 40 \\ 0 + 16 - 16 & 0 - 8 + 8 & 0 + 8 - 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,  $(\text{adj } A) A = |A| I = A (\text{adj } A)$

(iv) Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

Therefore cofactors of A

$$C_{11} = 3$$

$$C_{21} = -1$$

$$C_{31} = 1$$

$$C_{12} = -15$$

$$C_{22} = 7$$

$$C_{32} = -5$$

$$C_{13} = 4$$

$$C_{23} = -2$$

$$C_{33} = 2$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & -15 & 4 \\ -1 & 7 & -2 \\ 1 & -5 & 2 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A) A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 - 5 + 1 & 0 - 1 + 1 & -3 + 0 + 3 \\ -30 + 35 - 5 & 0 + 7 - 5 & 15 - 0 - 15 \\ 8 - 10 + 2 & 0 - 2 + 2 & -4 - 0 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Also, } |A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= [2(3 - 0) + 0(15 - 0) - 1(5 - 1)] \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (6 - 4) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Then, } A(\text{adj } A) = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 + 0 - 4 & -2 + 0 + 2 & 2 - 0 - 2 \\ 15 - 15 + 0 & -5 + 7 + 0 & 5 - 5 + 0 \\ 3 - 15 + 12 & -1 + 7 - 6 & 1 - 5 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence,  $(\text{adj } A)A = |A|I = A(\text{adj } A)$

3. For the matrix  $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$ , show that  $A(\text{adj } A) = 0$

Solution:

Given

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$$

Therefore cofactors of A

$$C_{11} = 30$$

$$C_{21} = 12$$

$$C_{31} = -3$$

$$C_{12} = -20$$

$$C_{22} = -8$$

$$C_{32} = 2$$

$$C_{13} = -50$$

$$C_{23} = -20$$

$$C_{33} = 5$$

We know that  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

By substituting these values in above matrix we get,

$$= \begin{bmatrix} 30 & -20 & -50 \\ 12 & 8 & 20 \\ -3 & 2 & 5 \end{bmatrix}^T$$

$$\text{So, adj } (A) = \begin{bmatrix} 30 & 12 & -3 \\ -20 & 8 & 2 \\ -50 & 20 & 5 \end{bmatrix}$$

$$\text{Now, } A (\text{adj } A) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix} \begin{bmatrix} 30 & 12 & -3 \\ -20 & 8 & 2 \\ -50 & -20 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 30 + 20 - 50 & 12 + 8 - 20 & -3 - 2 + 5 \\ 60 - 60 + 0 & 24 - 24 + 0 & -6 + 6 + 0 \\ 540 - 40 - 500 & 216 - 16 - 200 & -54 + 4 + 50 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,  $A [\text{adj } A] = 0$

4. If  $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ , show that  $\text{adj } A = A$

Solution:

Given

$$A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$

Cofactors of A

$$C_{11} = -4$$

$$C_{21} = -3$$

$$C_{31} = -3$$

$$C_{12} = 1$$

$$C_{22} = 0$$

$$C_{32} = 1$$

$$C_{13} = 4$$

$$C_{23} = 4$$

$$C_{33} = 3$$

We know that  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}^T$$

So,  $\text{adj } A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$

Hence,  $\text{adj } A = A$

5. If  $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ , show that  $\text{adj } A = 3A^T$ .

Solution:

Given

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Cofactors of A are

$$C_{11} = -3$$

$$C_{21} = 6$$

$$C_{31} = 6$$

$$C_{12} = -6$$

$$C_{22} = 3$$

$$C_{32} = -6$$

$$C_{13} = -6$$

$$C_{23} = 6$$

$$C_{33} = 3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$$

$$\text{So, adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$\text{Now, } 3A^T = 3 \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

Hence,  $\text{adj } A = 3A^T$

6. Find  $A(\text{adj}A)$  for the matrix  $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$

Solution:

Given

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$$

Cofactors of A are

$$C_{11} = 9$$

$$C_{12} = 19$$

$$C_{13} = -4$$

$$C_{21} = 4$$

$$C_{22} = 14$$

$$C_{23} = 1$$

$$C_{31} = 8$$

$$C_{32} = 3$$

$$C_{33} = 2$$

We know that  $\text{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}^T$$

$$\text{So, adj} A = \begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}$$

$$\text{Now, } A \text{ adj} A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix} \begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 8 + 24 & 19 - 28 + 9 & -4 - 2 + 6 \\ 0 + 8 - 8 & 0 + 28 - 3 & 0 + 2 - 2 \\ -36 + 20 + 16 & -76 + 70 + 6 & 16 + 5 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

Hence,  $A \text{adj } A = 25 I_3$

7. Find the inverse of each of the following matrices:

(i)  $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

(ii)  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iii)  $\begin{bmatrix} a & b \\ c & \frac{1+b^2}{a} \end{bmatrix}$

(iv)  $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

**Solution:**

(i) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

Now,  $|A| = \cos \theta (\cos \theta) + \sin \theta (\sin \theta)$

$= 1$

Hence,  $A^{-1}$  exists.

Cofactors of A are

$C_{11} = \cos \theta$

$C_{12} = \sin \theta$

$C_{21} = -\sin \theta$

$C_{22} = \cos \theta$

Since,  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

$(\text{adj } A) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^T$

$= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Now,  $A^{-1} = \frac{1}{|A|} \text{adj } A$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

(i) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

Now,  $|A| = -1 \neq 0$

Hence,  $A^{-1}$  exists.

Cofactors of A are

$$C_{11} = 0$$

$$C_{12} = -1$$

$$C_{21} = -1$$

$$C_{22} = 0$$

Since,  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

$$\text{adj } A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Now,  $A^{-1} = \frac{1}{|A|} \text{adj } A$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(iii) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$$\text{Now, } |A| = \frac{a+abc}{a} - bc = \frac{a+abc-abc}{a} = 1 \neq 0$$

Hence,  $A^{-1}$  exists.

Cofactors of  $A$  are

$$C_{11} = \frac{1+bc}{a}$$

$$C_{12} = -c$$

$$C_{21} = -b$$

$$C_{22} = a$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$\begin{aligned} |\text{adj } A| &= \begin{vmatrix} \frac{1+bc}{a} & -c \\ -b & a \end{vmatrix}^T \\ &= \begin{vmatrix} \frac{1+bc}{a} & -c \\ -b & a \end{vmatrix} \end{aligned}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} \frac{1+bc}{a} & -c \\ -b & a \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1+bc}{a} & -c \\ -b & a \end{bmatrix}$$

(iv) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$$\text{Now, } |A| = 2 + 15 = 17 \neq 0$$

Hence,  $A^{-1}$  exists.

Cofactors of  $A$  are

$$C_{11} = 1$$

$$C_{12} = 3$$

$$C_{21} = -5$$

$$C_{22} = 2$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$\text{[adj } A] = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$A^{-1} = \frac{1}{17} \begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$$

8. Find the inverse of each of the following matrices.

(i)  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

(ii)  $\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

(iii)  $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(iv)  $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(v)  $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

(vi)  $\begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$

$$(vii) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

Solution:

(i) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$$|A| = 1 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 1(6-1) - 2(4-3) + 3(2-9)$$

$$= 5 - 2 - 21$$

$$= -18 \neq 0$$

Hence,  $A^{-1}$  exists

Cofactors of A are

$$C_{11} = 5$$

$$C_{21} = -1$$

$$C_{31} = -7$$

$$C_{12} = -1$$

$$C_{22} = 7$$

$$C_{32} = 5$$

$$C_{13} = -7$$

$$C_{23} = 5$$

$$C_{33} = -1$$

We know that  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}^T$$

So,  $\text{adj } A = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}$

Now,  $A^{-1} = \frac{1}{|A|} \text{adj } A$

$$\text{So, } A^{-1} = \frac{1}{(-18)} \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{bmatrix}$$

Hence,

(ii) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$$|A| = 1 \begin{vmatrix} -1 & -1 \\ 3 & -1 \end{vmatrix} - 7 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 5 \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 1(1+3) - 2(-1+2) + 5(3+2)$$

$$= 4 - 2 + 25$$

$$= 27 \neq 0$$

Hence,  $A^{-1}$  exists

Cofactors of A are

$$C_{11} = 4$$

$$C_{21} = 17$$

$$C_{31} = 3$$

$$C_{12} = -1$$

$$C_{22} = -11$$

$$C_{32} = 6$$

$$C_{13} = 5$$

$$C_{23} = 1$$

$$C_{33} = -3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & -1 & 5 \\ 17 & -11 & 1 \\ 3 & 6 & -3 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{(27)} \begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} \frac{4}{27} & \frac{17}{27} & \frac{3}{27} \\ \frac{-1}{27} & \frac{-11}{27} & \frac{6}{27} \\ \frac{5}{27} & \frac{1}{27} & \frac{-3}{27} \end{bmatrix} = \begin{bmatrix} \frac{4}{27} & \frac{17}{27} & \frac{1}{9} \\ \frac{-1}{27} & \frac{-11}{27} & \frac{2}{9} \\ \frac{5}{27} & \frac{1}{27} & \frac{-1}{9} \end{bmatrix}$$

(III) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$$|A| = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 2(4 - 1) + 1(-2 + 1) + 1(1 - 2)$$

$$= 6 - 2$$

$$= 4 \neq 0$$

Hence,  $A^{-1}$  exists

Cofactors of A are

$$C_{11} = 3$$

$$C_{21} = 1$$

$$C_{31} = -1$$

$$C_{12} = +1$$

$$C_{22} = 3$$

$$C_{32} = 1$$

$$C_{13} = -1$$

$$C_{23} = 1$$

$$C_{21} = 3$$

$$\text{We know that } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}^T$$

$$\text{So, } \text{adj } A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

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(iv) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$$|A| = 2 \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} - 1 \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 2(3-0) - 0 - 1(5)$$

$$= 6 - 5$$

$$= 1 \neq 0$$

Hence,  $A^{-1}$  exists

Cofactors of A are

$$C_{11} = 3$$

$$C_{12} = -1$$

$$C_{13} = 1$$

$$C_{21} = -15$$

$$C_{21} = 6$$

$$C_{22} = -5$$

$$C_{12} = 5$$

$$C_{23} = -2$$

$$C_{32} = 2$$

$$\text{We know that } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}^T$$

$$\text{So, } \text{adj } A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{-1} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ 15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

(v) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$$|A| = \begin{vmatrix} -3 & 0 \\ -3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 4 & 4 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 4 & -3 \\ 3 & -3 \end{vmatrix}$$

$$= 0 - 1(16 - 12) - 1(-12 + 9)$$

$$= -4 + 3$$

$$= -1 \neq 0$$

Hence,  $A^{-1}$  exists

Cofactors of A are

$$C_{11} = 0$$

$$C_{21} = -1$$

$$\begin{aligned}
 C_{41} &= 1 \\
 C_{42} &= -4 \\
 C_{43} &= 3 \\
 C_{51} &= -4 \\
 C_{52} &= -3 \\
 C_{53} &= 3 \\
 C_{61} &= -4
 \end{aligned}$$

$$\text{We know that } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -4 & -3 \\ -1 & 3 & 3 \\ 1 & -4 & -4 \end{bmatrix}^T$$

$$\text{So, } \text{adj } A = \begin{bmatrix} 0 & -1 & 1 \\ -4 & 3 & -4 \\ -3 & 3 & -4 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 & 1 \\ -4 & 3 & -4 \\ -3 & 3 & -4 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

(vi) The criteria of existence of inverse matrix is the determinant of a given matrix should not equal to zero.

$$|A| = 0 \begin{vmatrix} 4 & 5 \\ -4 & -7 \end{vmatrix} - 0 \begin{vmatrix} 3 & 5 \\ -2 & -7 \end{vmatrix} - 1 \begin{vmatrix} 3 & 4 \\ -2 & -4 \end{vmatrix}$$

$$= 0 - 0 - 1(-12 + 8)$$

$$= 4 \neq 0$$

Hence,  $A^{-1}$  exists

Cofactors of A are

$$C_{11} = -8$$

$$C_{21} = 4$$

$$C_{22} = 4$$

$$C_{23} = 11$$

$$C_{31} = -2$$

$$C_{32} = -3$$

$$C_{33} = -4$$

$$C_{41} = 0$$

$$C_{42} = 0$$

$$C_{43} = 0$$

We know that  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} 8 & 11 & -4 \\ 4 & -2 & 0 \\ -4 & -3 & 0 \end{bmatrix}^T$$

So,  $\text{adj } A = \begin{bmatrix} 8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$

Now,  $A^{-1} = \frac{1}{|A|} \text{adj } A$

$$\text{So, } A^{-1} = \frac{1}{-4} \begin{bmatrix} 8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ \frac{11}{4} & \frac{-1}{2} & \frac{-3}{4} \\ -1 & 0 & 0 \end{bmatrix}$$

(vii) The criteria of existence of Inverse matrix is the determinant of a given matrix should not equal to zero.

$$|A| = 1 \begin{vmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{vmatrix} = 0 \neq 0$$

$$= -(\cos^2 \alpha - \sin^2 \alpha)$$

$$= -1 \neq 0$$

Hence,  $A^{-1}$  exists

Cofactors of A are

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$$C_{11} = -1$$

$$C_{21} = 0$$

$$C_{31} = 0$$

$$C_{12} = 0$$

$$C_{22} = -\cos \alpha$$

$$C_{32} = -\sin \alpha$$

$$C_{13} = 0$$

$$C_{23} = -\sin \alpha$$

$$C_{33} = \cos \alpha$$

$$\text{We know that } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}^T$$

$$\text{So, } \text{adj } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

9. Find the inverse of each of the following matrices and verify that  $A^{-1}A = I_3$ .

$$(i) \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

Solution:

(i) We have

$$|A| = 1 \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3 \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= 1(16 - 9) - 3(4 - 3) + 3(3 - 4)$$

$$= 7 - 3 - 3$$

$$= 1 \neq 0$$

Hence,  $A^{-1}$  exists

Cofactors of A are

$$C_{11} = 7$$

$$C_{21} = -3$$

$$C_{31} = -3$$

$$C_{12} = -1$$

$$C_{22} = 1$$

$$C_{32} = 0$$

$$C_{13} = -1$$

$$C_{23} = 0$$

$$C_{33} = 1$$

We know that  $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Also, } A^{-1}A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & 21-12-9 & 21-9-12 \\ -1+1+0 & -3+4+0 & -3+3+0 \\ -1+0+1 & -3+0+3 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,  $A^{-1}A = I_3$

(ii) We have

$$|A| = 2 \begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$$

$$= 2(8-7) - 3(6-3) + 1(21-12)$$

$$= 2 - 9 + 9$$

$$= 2 \neq 0$$

Hence,  $A^{-1}$  exists

Cofactors of A are

$$C_{11} = 1$$

$$C_{21} = 1$$

$$C_{31} = -1$$

$$C_{12} = -3$$

$$C_{22} = 1$$

$$C_{32} = 1$$

$$C_{13} = 9$$

$$C_{23} = -5$$

$$C_{33} = -1$$

**mc24 Myclass24**  
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$$\text{We know that } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{So, } \text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{Also, } A^{-1}A = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2+3-3 & 3+4-7 & 1+1-2 \\ -6+3+3 & -9+4+7 & -3+1+2 \\ 18-15-3 & 27-20-7 & 9-5-2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,  $A^{-1}A = I_3$

10. For the following pair of matrices verify that  $(AB)^{-1} = B^{-1}A^{-1}$ .

(i)  $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$

(ii)  $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

**Solution:**

(i) Given

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix},$$

$$|A| = 1 \neq 0$$

$$\text{Then, adj } A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix},$$

$$|B| = -10 \neq 0$$

$$\text{Then, adj } B = \begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj } B}{|B|} = -\frac{1}{10} \begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$$

$$\text{Also, } AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 12 + 6 & 18 + 4 \\ 28 + 15 & 42 + 10 \end{bmatrix}$$

$$AB = \begin{bmatrix} 18 & 22 \\ 43 & 52 \end{bmatrix}$$

$$|AB| = 936 - 946 = -10 \neq 0$$

$$\text{Adj } (AB) = \begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix}$$

$$(AB)^{-1} = \frac{\text{adj } AB}{|AB|} = \frac{1}{-10} \begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix} = \begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix}$$

$$\text{Now } B^{-1}A^{-1} = -\frac{1}{10} \begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$= -\frac{1}{10} \begin{bmatrix} 10 + 42 & -4 - 18 \\ -15 - 28 & 6 + 12 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix}$$

$$\text{Hence, } (AB)^{-1} = B^{-1}A^{-1}$$

(ii) Given

$$|A| = 1 \neq 0$$

$$\text{Adj } A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

$$|B| = 1 \neq 0$$

$$\text{adj } B = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj } B}{|B|} = \frac{1}{1} \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$$

$$\text{Also, } AB = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix}$$

$$|AB| = 407 - 406 = 1 \neq 0$$

$$\text{And, } \text{adj } (AB) = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$$

$$(AB)^{-1} = \frac{\text{adj } AB}{|AB|} = \frac{1}{1} \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$$

$$\text{Now, } B^{-1}A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$$

$$\text{Hence, } (AB)^{-1} = B^{-1}A^{-1}$$



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11. Let  $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ . Find  $(AB)^{-1}$

Solution:

Given

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

$$|A| = 15 - 14 = 1 \neq 0$$

$$\text{Therefore adj } A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

$$|B| = 54 - 56 = -2 \neq 0$$

$$\text{adj } B = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj } B}{|B|} = \frac{1}{-2} \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$$

$$\text{Now, } (AB)^{-1} = B^{-1}A^{-1}$$

$$= \frac{1}{-2} \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 45 + 49 & -18 - 21 \\ 40 - 42 & 16 + 18 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 94 & -39 \\ -2 & 34 \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} -47 & \frac{39}{2} \\ 1 & -17 \end{bmatrix}$$

12. Given  $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$ , compute  $A^{-1}$  and show that  $2A^{-1} = 9I - A$ .

Solution:

Given

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

$$|A| = 14 - 12 = 2 \neq 0$$

$$\text{adj } A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

To Show:  $2A^{-1} = 9I - A$

We have

$$\text{L.H.S} = 2A^{-1} = 2 \cdot \frac{1}{2} \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\text{R.H.S} = 9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

Hence,  $2A^{-1} = 9I - A$

13. If  $A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$ , then show that  $A - 3I = 2(I + 3A^{-1})$ .

Solution:

Given

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$|A| = 4 - 10 = -6 \neq 0$$

$$\text{adj } A = \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-6} \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$$

To Show:  $A - 3I - 2(I + 3A^{-1})$

We have

$$\text{LHS} = A - 3I$$

$$= \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$$

$$\text{RHS} = 2(I + 3A^{-1}) - 2I + 6A^{-1}$$

$$= 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 6 \frac{1}{-6} \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$$

Hence,  $A - 3I = 2(I + 3A^{-1})$

14. Find the inverse of the matrix  $A = \begin{bmatrix} a & b \\ c & \frac{b^2}{a} \end{bmatrix}$ , and show that  $aA^{-1} = (a^2 + bc + 1)I - aA$ .

**Solution:**

$$A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$$

$$\text{Now, } |A| = \frac{a+abc}{a} - bc = \frac{a+abc-abc}{a} = 1 \neq 0$$

Hence,  $A^{-1}$  exists.

Cofactors of A are

$$C_{11} = \frac{1+bc}{a}$$

$$C_{12} = -c$$

$$C_{21} = -b$$

$$C_{22} = a$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$\text{Adj } A = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}^T$$

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$$= \begin{bmatrix} 1+bc & -b \\ a & -c \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 1+bc & -b \\ a & -c \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1+bc & -b \\ a & -c \end{bmatrix}$$

To show  $aA^{-1} = (a^2 + bc + 1)I - aA$ .

$$\text{LHS} = aA^{-1}$$

$$= a \begin{bmatrix} 1+bc & -b \\ a & -c \end{bmatrix}$$

$$= \begin{bmatrix} a(1+bc) & -ab \\ a^2 & -ac \end{bmatrix}$$

$$\text{RHS} = (a^2 + bc + 1)I - aA$$

$$= \begin{bmatrix} a^2 + bc + 1 & 0 \\ 0 & a^2 + bc + 1 \end{bmatrix} - \begin{bmatrix} a^2 & ab \\ ac & 1+bc \end{bmatrix} = \begin{bmatrix} 1+bc & -ab \\ -ac & a^2 \end{bmatrix}$$

Hence, LHS = RHS

15. Given  $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ ,  $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ . Compute  $(AB)^{-1}$

Solution:

Given

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\text{Here, } (AB)^{-1} = B^{-1}A^{-1}$$

$$|A| = -5 + 4 = -1$$

Cofactors of A are

$$C_{11} = -1$$

$$C_{21} = 8$$

$$C_{31} = -12$$

$$C_{12} = 0$$

$$C_{22} = 1$$

$$C_{32} = -2$$

$$C_{13} = 1$$

$$C_{23} = -10$$

$$C_{33} = 15$$

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 8 & 1 & -10 \\ -12 & 2 & 15 \end{bmatrix}^T$$

$$\text{So, } \text{adj } A = \begin{bmatrix} 1 & 8 & -12 \\ 0 & 1 & 2 \\ -1 & -10 & 15 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1} \begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -8 & 12 \\ 0 & 1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 0 - 3 & -8 - 3 + 30 & 12 + 6 - 45 \\ 1 + 0 - 3 & -8 - 4 + 30 & 12 + 8 - 45 \\ 1 + 0 - 4 & -8 - 3 + 40 & 12 + 6 - 60 \end{bmatrix}$$

$$\text{Hence, } (AB)^{-1} = \begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$$

16. Let  $F(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $G(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$ . Show that

(i)  $[F(\alpha)]^{-1} = F(-\alpha)$

(ii)  $[G(\beta)]^{-1} = G(-\beta)$

(iii)  $[F(\alpha) G(\beta)]^{-1} = G(-\beta) F(-\alpha)$

Solution:

(i) Given

$$F(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|F(\alpha)| = \cos^2 \alpha + \sin^2 \alpha = 1 \neq 0$$

Cofactors of A are

$$C_{11} = \cos \alpha$$

$$C_{21} = \sin \alpha$$

$$C_{31} = 0$$

$$C_{12} = -\sin \alpha$$

$$C_{22} = \cos \alpha$$

$$C_{32} = 0$$

$$C_{13} = 0$$

$$C_{23} = 0$$

$$C_{33} = 1$$

$$\text{Adj } F(\alpha) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$\text{So, adj } F(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Now, } [F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|} \text{adj } F(\alpha) = \frac{1}{1} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \dots \text{ (i)}$$

$$\text{And, } F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \dots \dots \text{(ii)}$$

$$\text{Hence, } [F(\alpha)]^{-1} = F(-\alpha)$$

(ii) We have

$$|G(\beta)| = \cos^2 \beta + \sin^2 \beta = 1$$

Cofactors of A are

$$C_{11} = \cos \beta$$

$$C_{21} = 0$$

$$C_{31} = -\sin \beta$$

$$C_{12} = 0$$

$$C_{22} = 1$$

$$C_{32} = 0$$

$$C_{13} = \sin \beta$$

$$C_{23} = 0$$

$$C_{33} = \cos \beta$$

$$\text{Adj } G(\beta) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}^T$$

$$\text{So, adj } G(\beta) = \begin{bmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$\text{Now, } [G(\beta)]^{-1} = \frac{1}{1} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix} \dots \dots \text{(i)}$$

$$\text{And, } G(-\beta) = \begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ \sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \dots \text{[ii]}$$

$$\text{Hence, } [G(\beta)]^{-1} = G(-\beta)$$

(iii) Now we have to show that

$$*F(\alpha) G(\beta)^{-1} = G(-\beta) F(-\alpha)$$

We have already know that

$$*G(\beta)^{-1} = G(-\beta)$$

$$*F(\alpha)^{-1} = F(-\alpha)$$

$$\text{And LHS} = *F(\alpha) G(\beta)^{-1}$$

$$= *G(\beta)^{-1} *F(\alpha)^{-1}$$

$$= G(-\beta) F(-\alpha)$$

$$\text{Hence} = \text{RHS}$$

17. If  $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$  verify that  $A^2 - 4A + I = O$ , where  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . Hence find  $A^{-1}$ .

**Solution:**

Consider,

$$A^2 = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 4+3 & 6+6 \\ 2+2 & 3+4 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 12 \\ 4 & 7 \end{vmatrix}$$

$$4A = 4 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Now, } A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-8+1 & 12-2+0 \\ 4-4+0 & 7-8+1 \end{bmatrix}$$

$$\text{Hence, } = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Now, } A^2 - 4A + I = 0$$

$$A \cdot A - 4A = -I$$

Multiply by  $A^{-1}$  both sides we get

$$A \cdot A (A^{-1}) - 4A A^{-1} = -(A^{-1})$$

$$A I - 4I = -A^{-1}$$

$$A^{-1} + 4I - A I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

18. Show that  $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$  satisfies the equation  $A^2 + 4A - 42I = 0$ . Hence find  $A^{-1}$ .

**Solution:**

Given

$$A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 64 + 10 & -40 + 20 \\ -16 + 8 & 10 + 16 \end{bmatrix}$$

$$= \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix}$$

$$4A - 4I = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -32 & 20 \\ 8 & 16 \end{bmatrix}$$

$$42I = 42 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$$

Now,

$$A^2 + 4A - 42I = \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix} + \begin{bmatrix} -32 & 20 \\ 8 & 16 \end{bmatrix} - \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$$

$$= \begin{bmatrix} 74 - 32 & -20 + 20 \\ -8 + 8 & 26 + 16 - 42 \end{bmatrix}$$

Hence,  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now,  $A^2 + 4A - 42I = 0$

$$= A^{-1} A^2 + 4A^{-1} A - 42A^{-1} I = 0$$

$$= (A + 4I) - 42A^{-1} = 0$$

$$= 42A^{-1} = (A + 4I)$$

$$= A^{-1} \cdot \frac{1}{42} (A + 4I)$$

$$= \frac{1}{42} \left[ \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right]$$

$$= A^{-1} \cdot \frac{1}{42} \begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$$

19. If  $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$  show that  $A^3 - 5A + 7I = 0$ . Hence find  $A^{-1}$ .

Solution:

Given

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = \begin{vmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{vmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\text{Now, } A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,  $A^2 - 5A + 7I = 0$

Multiply by  $A^{-1}$  both sides

$$A \cdot A^{-1} A - 5A \cdot A^{-1} + 7I \cdot A^{-1} = 0$$

$$A - 5I + 7A^{-1} = 0$$

$$A^{-1} = \frac{1}{7} [5I - A]$$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$