NCERT Solutions for Class-XII Math

Chapter-5
Exercise- Miscellaneous

NCERT Math Class 12

1. Differentiate w.r.t. x the function $(3x^2 - 9x + 5)^9$

1. Let $y = \left(3x^2 - 9x + 5\right)^9$, therefore,

$$\frac{dy}{dx} = 9\left(3x^2 - 9x + 5\right)^8 \cdot \frac{d}{dx}\left(3x^2 - 9x + 5\right) = 9\left(3x^2 - 9x + 5\right)^8 \cdot (6x - 9)$$

$$= 27\left(3x^2 - 9x + 5\right)^8 \cdot (2x - 3)$$

2. Differentiate w.r.t. x the function $\sin^3 x + \cos^6 x$

2. Let $y = \sin^3 x + \cos^6 x$

Differentiating both sides with respect to x

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin^3 x) + \frac{d}{dx}\left(\cos^6 x\right) \quad \because \frac{d}{dx}(\sin x) = \cos x \ \& \ \frac{d}{dx}(\cos x) = -\sin x]$$

$$= 3\sin^2 x \times \frac{d}{dx}(\sin x) + 6\cos^5 x \times \frac{d}{dx}(\cos x)$$

$= 3\sin^2 x \times \cos x + 6\cos^5 x \times (-\sin x)$

$= 3 \sin x \cos x (\sin x - 2\cos^4 x)$

$$\therefore \frac{dy}{dx} = 3 \sin x \cos x (\sin x - 2\cos^4 x)$$

3. Differentiate w.r.t. x the function $(5x)^{3\cos 2x}$

3. Let $y = (5x)^{3\cos 2x}$, taking log on both sides

$$\log y = \log(5x)^{3\cos 2x} = 3\cos 2x \cdot \log 5x$$

Therefore, $\frac{1}{y}\frac{dy}{dx} = 3\cos 2x \cdot \frac{d}{dx}\log 5x + \log 5x \cdot \frac{d}{dx}3\cos 2x$

$$\Rightarrow \frac{dy}{dx} = y\left[3\cos 2x \cdot \frac{1}{5x} \cdot 5 + \log 5x \cdot 3(-\sin 2x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3(5x)^{3\cos 2x}\left[\frac{\cos 2x - 2\sin 2x \log 5x}{x}\right]$$

4. Differentiate w.r.t. x the function $\sin^{-1}\left(x\sqrt{x}\right),\ 0 \le x \le 1$

4. Let y = $sin^{-1}\left(x\sqrt{x}\right)$, $0 \le x \le 1$

Differentiating both sides with respect to x, we get
Using chain rule we get

$$\frac{dy}{dx}=\frac{d}{dx}sin^{-1}\left(x\sqrt{x}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}}\times\frac{d}{dx}\left(x\sqrt{x}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^3}}\times\frac{d}{dx}\left(x^{\frac{3}{2}}\right)=\frac{1}{\sqrt{1-x^3}}\times\frac{3}{2}x^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

$$\therefore \frac{dy}{dx}=\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

5. Differentiate w.r.t. x the function $\frac{\cos^{-1}}{\sqrt{2x+7}}, -2<x<2$

5. Let $y=\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$, therefore,

$$\frac{dy}{dx}=\frac{\cos^{-1}\frac{x}{2}\cdot\frac{d}{dx}\sqrt{2x+7}-\sqrt{2x+7}\frac{d}{dx}\cos^{-1}\frac{x}{2}}{(\sqrt{2x+7})^2}$$

$$=\frac{\left[\cos^{-1}\frac{x}{2}\cdot\frac{1}{2\sqrt{2x+7}}\cdot 2\right]-\sqrt{2x+7}\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2}}{2x+7}$$

$$=\frac{\cos^{-1}\frac{x}{2}\cdot\frac{1}{\sqrt{2x+7}}+\sqrt{2x+7}\frac{1}{\sqrt{4-(x)^2}}}{2x+7}=\frac{\cos^{-1}\frac{x}{2}\cdot\sqrt{4-x^2}+2x+7}{\left(2x+7\right)\sqrt{2x+7}\sqrt{4-x^2}}$$

6. Differentiate w.r.t. x the function $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), 0<x<\frac{\pi}{2}$

6. Let y = $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), 0<x<\dfrac{\pi}{2}$

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)}$$

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)(1-\sin x)}$$

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\frac{2+2\sqrt{1-sin^2 x}}{2\sin x}$$

$$\Rightarrow \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\frac{1+\cos x}{\sin x}=\frac{2cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\cot\frac{x}{2}$$

Substituting the value of $\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\cot\dfrac{x}{2}$ in y.

$\therefore$ y = $cot^{-1}\left(\cot\dfrac{x}{2}\right)$

$\Rightarrow$ y = x/2

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}$$

7. Differentiate w.r.t. x the function $(\log x)^{\log x}, x>1$

7. Let $y=(\log x)^{\log x}$, taking log on both sides

$$\log y=\log(\log x)^{\log x}=\log x\cdot\log(\log x)$$

Therefore,

$$\frac{1}{y}\frac{dy}{dx}=\log x\cdot\frac{d}{dx}\log(\log x)+\log(\log x)\cdot\frac{d}{dx}\log x$$

$$\Rightarrow \frac{dy}{dx}=y\left[\log x\cdot\frac{1}{\log x}\cdot\frac{1}{x}+\log(\log x)\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{dy}{dx}=(\log x)^{\log x}\left[\frac{1+\log(\log x)}{x}\right]$$

8. Differentiate w.r.t. x the function cos (a cos x + b sin x), for some constant a and b.

8. Let y = cos (a cos x + b sin x)

a and b are some constants

y = cos (a cos x + b sin x)

Differentiating both sides with respect to x, we get

Using chain rule

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\cos(\text{a} \cos x + b \sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x) \times \frac{d}{dx}(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x) \times \left[a(-\sin x) + b\cos x\right]$$

$$\therefore \frac{dy}{dx} = (a\sin x - b\cos x) \times \sin(a\cos x + b\sin x)$$

9. Differentiate w.r.t. x the function $(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

9. Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, taking log on both sides

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x) \cdot \frac{d}{dx}\log(\sin x - \cos x) + \log(\sin x - \cos x) \cdot$$

$$\frac{d}{dx}(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = y$$

$$\left[(\sin x - \cos x) \cdot \frac{(\cos x + \sin x)}{(\sin x - \cos x)} + \log(\sin x - \cos x)(\cos x + \sin x)\right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)\left[1 + \log(\cos x - \sin x)\right]$$

10. Differentiate w.r.t. x the function $x^x + x^a + a^x + a^a$, for some fixed a > 0 and x > 0

10. Let $y = x^x + x^a + a^x + a^a$, for some fixed a > 0 and x > 0

And let $x^x = u$, $x^a = v$, $a^x = w$ and $a^a = s$

Then y = u + v + w + s

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \qquad \text{.............................(I)}$$

Now,

$u = x^x$

Taking logarithm both sides, we get

$\log u = \log x^x$

$\Rightarrow$ log u = x log x

Differentiating both sides w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = \log x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x + x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1 + \log x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}$$

$v = x^a$

Differentiating both sides with respect to x

$$\frac{dv}{dx} = \frac{d}{dx}\left(x^a\right)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III)}$$

$w = a^x$

Taking logarithm both sides

log w = log a^x

log w = x log a

Differentiating both sides with respect to x

$$\frac{1}{w}\frac{dw}{dx} = \log a \times \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(IV)}$$

$s = a^a$

Differentiating both sides with respect to x

$$\frac{ds}{dx} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(V)}$$

Putting (II), (III), (IV) and (V) in (I)

$$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a + 0$$

$$\therefore \frac{dy}{dx} = x^x(1 + \log x) + a\,x^{a-1} + a^x \log a$$

11. Differentiate w.r.t. x the function $x^{x^2-3} + (x-3)^{x^2}$, for x > 3

11. Let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$ therefore, $y = u + v$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

Here, $u=x^{x^2-3}$, taking log on both sides

$\log u=\left(x^2-3\right)\log x$, therefore,

$$\frac{1}{u}\frac{du}{dx}=\left(x^2-3\right)\cdot\frac{d}{dx}\log x+\log x\cdot\frac{d}{dx}\left(x^2-3\right)$$

$$=\left(x^2-3\right)\cdot\frac{1}{x}+\log x\cdot 2x$$

$$\frac{du}{dx}=u\left[\frac{x^2-3+2x^2\log x}{x}\right]$$

$$\frac{du}{dx}=x^{x^2-3}\left[\frac{x^2-3+2x^2\log x}{x}\right]=x^{x^2-4}\left(x^2-3+2x^2\log x\right)\ldots$$

and, $v=(x-3)^{x^2}$, taking log on both sides $\log v=x^2\log\left(x-3\right)$, therefore,

$$\frac{1}{v}\frac{dv}{dx}=x^2\cdot\frac{d}{dx}\log\left(x-3\right)+\log\left(x-3\right)\cdot\frac{d}{dx}x^2$$

$$=x^2\cdot\frac{1}{x-3}+\log\left(x-3\right)\cdot 2x=\frac{x^2}{x-3}+2x\cdot\log\left(x-3\right)$$

$$\frac{dv}{dx}=v\left[\frac{x^2}{x-3}+2x\cdot\log\left(x-3\right)\right]=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\cdot\log\left(x-3\right)\right]\ldots(3)$$

Putting the value of $\frac{du}{dx}$ from (2) and value of $\frac{dv}{dx}$ from (3) in equation (1), we have

$$\frac{dy}{dx}=x^{x^2-4}\left(x^2-3+2x^2\log x\right)+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\cdot\log\left(x-3\right)\right]$$

12. Find $\frac{dy}{dx}$, if y = 12 (1 – cos t), x = 10 (t – sin t), $-\frac{\pi}{2}<t<\frac{\pi}{2}$

12. To find $\frac{dy}{dx}$ we need to find out $\frac{dx}{dt}$ and $\frac{dy}{dt}$

So, $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given, y = 12 (1 – cos t) and x = 10 (t – sin t)

x = 10 (t – sin t)

Differentiating with respect to t.

$$\frac{dx}{dt}=\frac{d}{dt}\left[10\left(t-\sin t\right)\right]$$

$$\Rightarrow \frac{dx}{dt} = 10 \times \frac{d}{dt}(t - \sin t) = 10(1 - \cos t)$$

y = 12 (1 – cos t)

Differentiating with respect to t.

$$\frac{dy}{dt} = \frac{d}{dt}\left[12(1 - \cos t)\right]$$

$$\Rightarrow \frac{dy}{dt} = 12 \times \frac{d}{dt}(1 - \cos t) = 12 \times \left[0 - (-\sin t)\right] = 12 \sin t$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10(1 - \cos t)} = \frac{12 \times 2 \sin\frac{t}{2}\cos\frac{t}{2}}{10 \times 2 sin^2 \frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$$

$$\therefore \frac{dy}{dx} = \frac{6}{5}\cot\frac{t}{2}$$

13. Find $\frac{dy}{dx}$, if $y = sin^{-1}x + sin^{-1}\sqrt{1 - x^2}$, 0 < x < 1

13. Here, $y = \sin^{-1}x + \sin^{-1}\sqrt{1 - x^2}$, therefore

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}x + \frac{d}{dx}\sin^{-1}\sqrt{1 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - \left(\sqrt{1 - x^2}\right)^2}}\frac{d}{dx}\sqrt{1 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{x}\cdot\frac{1}{2\sqrt{1 - x^2}}\frac{d}{dx}(1 - x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{x}\cdot\frac{1}{2\sqrt{1 - x^2}}(-2x) = \frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - x^2}} = 0$$

14. If $x\sqrt{1 + y} + y\sqrt{1 + x} = 0$, for , – 1 < x < 1, prove that $\frac{dy}{dx} = -\frac{1}{(1 + x)^2}$

14. Given, $x\sqrt{1 + y} + y\sqrt{1 + x} = 0$

$$x\sqrt{1 + y} = -y\sqrt{1 + x}$$

Now, squaring both sides, we get

$$\Rightarrow \left(x\sqrt{1 + y}\right)^2 = \left(-y\sqrt{1 + x}\right)^2$$

$$\Rightarrow x^2(1 + y) = y^2(1 + x)$$

$\Rightarrow x^2 + x^2y = y^2 + y^2x$

$\Rightarrow x^2 - y^2 = xy^2 - x^2y$

$\Rightarrow$ (x + y)(x – y) = xy (y – x)

$\Rightarrow$ x + y = –xy

$\Rightarrow$ y + xy = –x

$\Rightarrow$ y (1 + x) = –x

$\Rightarrow \; y=-\frac{x}{(1+x)}$

Differentiating both sides with respect to x, we get

$y=-\frac{x}{(1+x)}$

Using Quotient Rule

$$\frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}=-\frac{(1+x)-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

$\therefore \; \frac{dy}{dx}=-\frac{1}{(1+x)^2}$

Hence, Proved

15. If $(x-a)^2+(y-b)^2=c^2$, for some $c>0$, prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

15. Given that: $(x-a)^2+(y-b)^2=c^2$

Differentiating with respect to x, we have

$$\frac{d}{dx}(x-a)^2+\frac{d}{dx}(y-b)^2=\frac{d}{dx}c^2 \Rightarrow 2(x-a)+2(y-b)\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x-a}{y-b}$$

Differentiating again, we have

$$\frac{d^2y}{dx^2}=-\frac{(y-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(y-b)}{(y-b)^2}=-\frac{(y-b)1-(x-a)\frac{dy}{dx}}{(y-b)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\frac{(y-b)1-(x-a)\left(-\frac{x-a}{y-b}\right)}{(y-b)^2}=-\frac{(y-b)^2+(x-a)^2}{(y-b)^3}=-\frac{c^2}{(y-b)^3}$$

Putting the values in $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$, we have

$$\frac{\left[1+\left(-\frac{x-a}{y-b}\right)^2\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}=\frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$$=\frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}=\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$=\frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}=-\frac{c^3}{c^2}=-c$, which is a constant independent of a and b.

16. If cos y = x cos (a + y), with cos a ≠ ± 1, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

16. Given, cos y = x cos (a + y)

Differentiating both sides with respect to x

$$\frac{d}{dy}[\cos y]=\frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx}=\cos(a+y)\times\frac{d}{dx}(x)+x\times\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx}=\cos(a+y)+x[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y)-\sin y]\frac{dy}{dx}=\cos(a+y) \qquad \text{...........................(I)}$$

Since, cos y = x cos (a + y) ⇒ x = cos y/cos (a + y)

Substituting the value of x in (I)

$$\left[\frac{\cos y}{\cos(a+y)}\times\sin(a+y)-\sin y\right]\frac{dy}{dx}=\cos(a+y)$$

$$\Rightarrow [\cos y\times\sin(a+y)-\sin y\times\cos(a+y)]\frac{dy}{dx}=\cos(a+y)\times\cos(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = cos^2(a+y)$$

$$\Rightarrow \sin a \times \frac{dy}{dx} = cos^2(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{cos^2(a+y)}{\sin a}$$

Hence, proved

17. If x = a (cos t + t sin t) and y = a (sin t – t cos t), find $\frac{d^2y}{dx^2}$

17. Here, $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$

Therefore,

$\frac{dx}{dt} = a[-\sin t + (t\cos t + \sin t)] = at\cos t$ and

$$\frac{dy}{dt} = a[(\cos t - (-t\sin t + \cos t))] = at\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{1}{at\cos t} = \frac{\sec^3 t}{at}$$

18. If f (x) = |x|³, show that f″(x) exists for all real x and find it.

18. $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$

When, x ≥ 0,

$f(x) = |x|^3 = x^3$

So, $f'(x) = 3x^2$

And $f''(x) = d(f'(x))/dx = 6x$

∴ $f''(x) = 6x$

When x < 0,

$f(x) = |x|^3 = (-x)^3 = -x^3$

$f'(x) = -3x^2$

$f''(x) = -6x$

$$\therefore f''(x) = \begin{cases} 6x, & x \geq 0 \\ -6x, & x < 0 \end{cases}$$

19. Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

19. Let, $P(n): \frac{d}{dx}\left(x^n\right) = nx^{n-1}$

Putting $n=1$, we have LHS $= \frac{d}{dx}\left(x^1\right) = 1$ and RHS $= 1x^{1-1} = x^0 = 1$

Hence, $P(n)$ is true for $n=1$.

Let, $P(k): \frac{d}{dx}\left(x^k\right) = kx^{k-1}$ is true.

To prove: $P(k+1): \frac{d}{dx}\left(x^{k+1}\right) = (k+1)x^k$ is also true.

LHS $= \frac{d}{dx}\left(x^{k+1}\right) = \frac{d}{dx}\left(x^k \cdot x\right) = x^k \frac{d}{dx}(x) + x\frac{d}{dx}x^k$

$= x^k \cdot 1 + x \cdot kx^{k-1} = (1+k)x^k = RHS$

Hence, $P(n)$ is true for $n = k+1$.

Therefore, by the principle of Mathematical Induction $P(n)$ is true for all natural numbers n.

20. Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

20. sin (A + B) = sin A cos B + cos A sin B

Differentiating with respect to x, we get

$$\frac{d}{dx}\left[\sin(A+B)\right] = \frac{d}{dx}(\sin A\cos B) + \frac{d}{dx}(\cos A\sin B)$$

$\Rightarrow$

$$\cos(A+B)\frac{d}{dx}(A+B) = \cos B\frac{d}{dx}(\sin A) + \sin A\frac{d}{dx}(\cos B) + \sin B\frac{d}{dx}(\cos A) + \cos A\frac{d}{dx}(\sin B)$$

$\Rightarrow$

$$\cos(A+B)\frac{d}{dx}(A+B) = \cos B\cos A\frac{dA}{dx} + \sin A\,(-\sin B)\frac{dB}{dx} + \sin B(-\sin A)\frac{dA}{dx} + \cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\times\left[\frac{dA}{dx} + \frac{dB}{dx}\right] = (\cos A\cos B - \sin A\sin B)\times\left[\frac{dA}{dx} + \frac{dB}{dx}\right]$$

$\therefore$ cos (A + B) = cos A cos B – sin A sin B

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

21. Function $f(x) = |x-1| + |x-3|$ is continuous for all real points but not differentiable at two points (x = 1 and x = 3)

22. If $\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

22. Let y = $\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Differentiation of determinant $u = \begin{vmatrix} e & f & g \\ h & i & j \\ k & l & m \end{vmatrix}$ is given by

$$\frac{du}{dx} = \begin{vmatrix} \frac{d}{dx}(e) & \frac{d}{dx}(f) & \frac{d}{dx}(g) \\ h & i & j \\ k & l & m \end{vmatrix} + \begin{vmatrix} e & f & g \\ \frac{d}{dx}(h) & \frac{d}{dx}(i) & \frac{d}{dx}(j) \\ k & l & m \end{vmatrix} + \begin{vmatrix} e & f & g \\ h & i & j \\ \frac{d}{dx}(k) & \frac{d}{dx}(l) & \frac{d}{dx}(m) \end{vmatrix}$$

∴

$$\frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}(f(x)) & \frac{d}{dx}(g(x)) & \frac{d}{dx}(h(x)) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \frac{d}{dx}(l) & \frac{d}{dx}(m) & \frac{d}{dx}(n) \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ \frac{d}{dx}(a) & \frac{d}{dx}(b) & \frac{d}{dx}(c) \end{vmatrix}$$

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

Since, a, b, c and l, m, n are constants so, their differentiation is zero.
Also in a determinant if all the elements of row or column turns to be zero then the value of determinant is zero.

$$\therefore \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} = 0 \ and \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\therefore \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Hence, proved.

23. If, $y = e^{a\cos^{-1}x}$ $-1 \le x \le 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

23. Given that: $y = e^{a\cos^{-1}x}$, therefore,

Differentiating with respect to x, we have

⇨ $\frac{dy}{dx} = \frac{d}{dx}e^{a\cos^{-1}x} = e^{a\cos^{-1}x}\frac{d}{dx}a\cos^{-1}x$

⇨ $\Rightarrow \frac{dy}{dx} = e^{a\cos^{-1}x}a \cdot \frac{-1}{\sqrt{1-x^2}} = -\frac{ay}{\sqrt{1-x^2}}$

Squaring both the sides, we have

$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2} \Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$

Differentiating again with respect to x, we have

$\left(1-x^2\right) \cdot 2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \frac{d}{dx}\left(1-x^2\right) = a^2 2y\frac{dy}{dx}$

⇨ $\Rightarrow \frac{dy}{dx}\left[2\left(1-x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x)\right] = 2a^2y\frac{dy}{dx}$

⇨ $\Rightarrow 2\frac{dy}{dx}\left[\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right] = 2a^2y\frac{dy}{dx}$

⇨ $\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$

⇨ $\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$