NCERT Solutions for Class-XI Maths

Chapter-9 Exercise-Miscellaneous

NCERT Math Class 11

1. Show that the sum of $(m+n)^{\text{th}}$ and $(m-n)^{\text{th}}$ terms of an A.P. is equal to twice the m^{th} term.

1. Let a and d be the first term and the common difference of the A.P. respectively. It is known that the k^{th} term of an A. P. is given by

$a_k = a+(k-1)d$

$\therefore a_{m+n} = a+(m+n-1)d$

$a_{m-n} = a+(m-n-1)d$

$a_m = a+(m-1)d$

$\therefore a_{m+n}+a_{m-n} = a+(m+n-1)d+a+(m-n-1)d$

$= 2a+(m+n-1+m-n-1)d$

$= 2a+(2m-2)d$

$= 2a+2(m-1)d$

$= 2\left[a+(m-1)d\right]$

$= 2a_m$

Thus, the sum of $(m+n)^{\text{th}}$ and $(m-n)^{\text{th}}$ terms of an A.P. is equal to twice the m^{th} term.

2. If the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.

2. Let the three numbers in A.P. be a – d, a, and a + d.

mAccording to question -

(a – d) + (a) + (a + d) = 24 … (1)

⇒ 3a = 24

∴ a = 8

and,

(a – d) a (a + d) = 440 … (2)

⇒ (8 – d) (8) (8 + d) = 440

⇒ (8 – d) (8 + d) = 66

⇒ 64 – d2 = 66

⇒ d2 = 64 – 66 = 9

⇒ d = ± 3

Therefore,

when d = 3, the numbers are 6, 8, and 11 and

when d = –3, the numbers are 11, 8, and 6.

Thus, the three numbers are 6, 8, and 11.

3. Let the sum of $n, 2n, 3n$ terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3 = 3(S_2 - S_1)$

3. Let a and b be the first term and the common difference of the A.P. respectively. Therefore,

$$S_1 = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_2 = \frac{2n}{2}\left[2a + (2n-1)d\right] = n\left[2a + (2n-1)d\right]$$

$$S_3 = \frac{3n}{2}\left[2a + (3n-1)d\right]$$

From (1) and (2), we obtain

$$S_2 - S_1 = n\left[2a + (2n-1)d\right] - \frac{n}{2}\left[2a + (n-1)d\right]$$

$$= n\left\{\frac{4a + 4nd - 2d - 2a - nd + d}{2}\right\}$$

$$= n\left[\frac{2a + 3nd - d]}{2}\right]$$

$$= \frac{n}{2}\left[2a + (3n-1)d\right]$$

$$\therefore 3(\ S_2 - S_1) = \frac{3n}{2}\left[2a + (3n-1)d\right] = S_3$$

Hence, the given result is proved.

4. Find the sum of all numbers between 200 and 400 which are divisible by 7.

4. The numbers lying between 200 and 400 which are divisible by 7

are as follows: -

203, 210, 217, … 399

Since the common difference between the consecutive terms is constant. Thus, the above sequence is an A.P.

∴First term, a = 203

Last term, $l = 399$

Common difference, $d = 7$

Let the number of terms of the A.P. be n.

$\therefore a_n = 399 = a + (n - 1) d$

$\Rightarrow 399 = 203 + (n - 1) 7$

$\Rightarrow 7 (n - 1) = 196$

$\Rightarrow n - 1 = 28$

$\Rightarrow n = 29$

We know that -

Sum of n terms of an A.P(S_n) = (n/2)[a + l]

$S_{29} = (29/2)[203 + 399]$

$= (29/2)[602]$

$= 29 \times 301$

$= 8729$

Thus, the required sum is 8729.

5. Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

5. The integers from 1 to 100 , which are divisible by 2 , are $2, 4, 6 \ldots 100$.

This forms an A.P. with both the first term and common difference equal to 2.

$$\Rightarrow 100 = 2 + (n - 1)2$$

$$\Rightarrow n = 50$$

$$\therefore 2 + 4 + 6 + \ldots + 100 = \frac{50}{2}\left[2(2) + (50 - 1)(2)\right]$$

$$= \frac{50}{2}[4 + 98]$$

$$= (25)(102)$$

$$= 2550$$

The integers from 1 to 100 , which are divisible by 5 , are $5, 10 \ldots 100$.

This forms an A.P. with both the first term and common difference equal to 5.

$$\therefore 100 = 5 + (n - 1)5$$

$$\Rightarrow 5n = 100$$

$$\Rightarrow n = 20$$

$$\therefore 5+10+\ldots+100=\frac{20}{2}\left[2(5)+(20-1)5\right]$$

$=10\left[10+(19)5\right]$

$=10[10+95]=10\times 105$

$=1050$

The integers, which are divisible by both 2 and 5 , are $10,20,\ldots 100$.

This also forms an A.P. with both the first term and common difference equal to 10.

$\therefore 100=10+(n-1)(10)$

$\Rightarrow 100=10n$

$\Rightarrow n=10$

$$\therefore 10+20+\ldots+100=\frac{10}{2}\left[2(10)+(10-1)(10)\right]$$

$=5[20+90]=5(110)=550$

$\therefore$ Required sum $=2550+1050-550=3050$

Thus, the sum of the integers from 1 to 100 , which are divisible by 2 or 5 , is 3050.

6. Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder.

6. The two - digit numbers which when divided by 4, yield 1 as remainder, are as follows: -

13, 17, … 97

Since the common difference between the consecutive terms is constant. Thus, the above sequence is an A.P. with first term 13 and common difference 4.

Last Term of the A.P(l) = 97

Let n be the number of terms of the A.P.

It is known that the nth term of an A.P. is given by -

$a_n = a + (n-1)\, d$

$\Rightarrow 97 = 13 + (n-1)\,(4)$

$\Rightarrow 4\,(n-1) = 84$

$\Rightarrow n - 1 = 21$

$\Rightarrow n = 22$

Sum of n terms of an A.P. is given by -

$S_n = (n/2)[a + l]$

$\therefore S_{22} = (22/2)[13 + 97]$

$= 11 \times 110$
$= 1210$
Thus, the required sum is 1210.

7. If f is a function satisfying $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in N$, such that $f(1) = 3$ and $\sum_{1}^{n} f(x) = 120$, find the value of n.

7. It is given that,
$f(x+y) = f(x) \times f(y)$ for all $x, y \in \mathrm{N}$
$f(1) = 3$

Taking $x = y = 1$ in (1),
we obtain $f(1+1) = f(2) = f(1)f(1) = 3 \times 3 = 9$
Similarly,
$f(1+1+1) = f(3) = f(1+2) = f(1)f(2) = 3 \times 9 = 27$

$f(4) = f(1+3) = f(1)f(3) = 3 \times 27 = 81$

$\therefore f(1), f(2), f(3), \ldots$, that is $3, 9, 27, \ldots$, forms a G.P. with both the first term and common ratio equal to 3.

It is known that, $S_n = \dfrac{a(r^n - 1)}{r - 1}$
It is given that, $\sum_{x=1}^{n} f(x) = 120$

$$\therefore 120 = \frac{3(3^n - 1)}{3 - 1}$$

$$\Rightarrow 120 = \frac{3}{2}(3^n - 1)$$

$\Rightarrow 3^n - 1 = 80$
$\Rightarrow 3^n = 81 = 3^4$
$\therefore n = 4$
Thus, the value of n is 4.

8. The sum of some terms of G.P. is 316 whose first term and the common ratio are 6 and 2, respectively. Find the last term and the number of terms.

8. Let the sum of n terms of the G.P. be 316.
It is known that,

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

It is given that the first term a is 6 and common ratio r is 2.

$$\therefore 316 = \frac{6(2^n - 1)}{(2-1)}$$

$$\Rightarrow 316 = 6(2^n - 1)$$

$$\Rightarrow \frac{316}{6} = 2^n - 1$$

$$\Rightarrow 63 = 2^n - 1$$

$$\Rightarrow 2^n = 64 = 2^6$$

$\therefore$ n = 6

nth term of G.P is given by -

$a_n = ar^{n-1}$

$\therefore$ Last term of the G.P = 6th term = $ar^{6-1} = (6)(2)^6 = (6)(32) = 160$

Thus, the last term of the G.P. is 160.

9. The first term of a G.P. is 1 . The sum of the third term and fifth term is 90 . Find the common ratio of G.P.

9. Let a and r be the first term and the common ratio of the G.P. respectively.

$\therefore a = 1$

$$a_3 = ar^2 = r^2$$

$$a_5 = ar^4 = r^4$$

$$\therefore r^2 + r^4 = 90$$

$$\Rightarrow r^4 + r^2 - 90 = 0$$

$$\Rightarrow r^2 = \frac{-1 + \sqrt{1+360}}{2} = \frac{-1 \pm \sqrt{361}}{2} = \frac{-1 \pm 19}{2} = -10 \text{ or } 9$$

$\therefore r = \pm 3$ (Taking real roots)

Thus, the common ratio of the G.P. is ± 3.

10. The sum of three numbers in G.P. is 66. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

10. Let the three numbers in G.P. be a, ar, and ar^2.

According to question -

$a + ar + ar^2 = 66$

$\Rightarrow a(1 + r + r^2) = 66$

$\Rightarrow a = \dfrac{66}{1 + r + r^2}$ …(1)

Given that -

$(a - 1), (ar - 7), (ar^2 - 21)$ forms an A.P.

thus the common difference between the consecutive terms will be equal.

$\therefore (ar - 7) - (a - 1) = (ar^2 - 21) - (ar - 7)$

$\Rightarrow ar - a - 6 = ar^2 - ar - 14$

$\Rightarrow ar^2 - 2ar + a = 8$

$\Rightarrow a(r^2 - 2r + 1) = 8$

$\Rightarrow a = \dfrac{8}{1 - 2r + r^2}$ …(2)

Comparing equations (1) & (2), we get -

$$\frac{66}{1 + r + r^2} = \frac{8}{1 - 2r + r^2}$$

On Cross - multiplying,

$\Rightarrow 66(r^2 - 2r + 1) = 8(1 + r + r^2)$

$\Rightarrow 7(r^2 - 2r + 1) = (1 + r + r^2)$

$\Rightarrow 7r^2 - 14r + 7 = 1 + r + r^2$

$\Rightarrow 6r^2 - 16r + 6 = 0$

$\Rightarrow 6r^2 - 12r - 3r + 6 = 0$

$\Rightarrow 6r (r - 2) - 3 (r - 2) = 0$

$\Rightarrow (6r - 3) (r - 2) = 0$

$\therefore r = 2$ or $r = 1/2$

When r = 2, a = 8

When r = 1/2, a = 32

Therefore,

when r = 2, the three numbers in G.P. are 8, 16, and 32.

when r = 1/2, the three numbers in G.P. are 32, 16, and 8.

Thus, in either case, the three required numbers are 8, 16, and 32.

11. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

11. Let the G.P. be $T_1, T_2, T_3, T_4 \ldots T_{2n}$.

Number of terms $= 2n$

According to the given condition,

$$T_1 + T_2 + T_3 + \ldots + T_{2n} = 5\left[T_1 + T_3 + \ldots + T_{2n-1}\right]$$

$$\Rightarrow T_1 + T_2 + T_3 + \ldots + T_{2n} - 5\left[T_1 + T_3 + \ldots + T_{2n-1}\right] = 0$$

$$\Rightarrow T_2 + T_4 + \ldots + T_{2n} = 4\left[T_1 + T_3 + \ldots + T_{2n-1}\right]$$

Let the G.P. be $a, ar, ar^2, ar^3 \ldots$

$$\therefore \frac{ar\left(r^n - 1\right)}{r-1} = \frac{4 \times a\left(r^n - 1\right)}{r-1}$$

$\Rightarrow ar = 4a$

$\Rightarrow r = 4$

Thus, the common ratio of the G.P. is 4 .

12. The sum of the first four terms of an A.P. is 66. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

12. Let the A.P. be a, a + d, a + 2d, a + 3d, ... a + (n – 2) d, a + (n – 1)d.

Sum of first four terms = a + (a + d) + (a + 2d) + (a + 3d) = 4a + 6d

Sum of last four terms = [a + (n – 4) d] + [a + (n – 3) d] + [a + (n – 2) d] + [a + (n – 1) d] = 4a + (4n – 10) d

According to question -

4a + 6d = 66

⇒ 4(11) + 6d = 66 [∵ a = 11 (given)]

⇒ 6d = 12

⇒ d = 2

and,

4a + (4n –10) d = 112

⇒ 4(11) + (4n – 10)2 = 112

⇒ (4n – 10)2 = 68

⇒ 4n – 10 = 34

⇒ 4n = 44

⇒ n = 11

Thus, the number of terms of the A.P. is 11.

13. If $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}(x\neq 0)$ then show that a,b,c and d are in G.P.

13. It is given that, $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}$

$\Rightarrow (a+bx)(b-cx)=(b+cx)(a-bx)$

$\Rightarrow ab-acx+b^2x-bcx^2=ab-b^2x+acx-bcx^2$

$\Rightarrow 2b^2x=2acx$

$\Rightarrow b^2=ac$

$\Rightarrow \frac{b}{a}=\frac{c}{b}$

Also, $\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$

$\Rightarrow (b+cx)(c-dx)=(b-cx)(c+dx)$

$\Rightarrow bc-bdx+c^2x-cdx^2=bc+bdx-c^2x-cdx^2$

$\Rightarrow 2c^2x=2bdx$

$\Rightarrow c^2=bd$

$\Rightarrow \frac{c}{d}=\frac{d}{c}$

From (1) and (2), we obtain

$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$

Thus, a,b,c, and d are in G.P.

14. Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n = S^n$.

14. Let the G.P. be a, ar, ar^2, ar^3, … ar^{n-1}…

According to question -

$S=\frac{a(r^n-1)}{(r-1)}$

$P=a^n\times r^{1+2+3+\ldots+(n-1)}$

$=a^n\times r^{\frac{n(n-1)}{2}}$ [∵ Sum of first n natural numbers is n(n + 1)/2]

$$R=\frac{1}{a}+\frac{1}{ar}+\ldots+\frac{1}{ar^{n-1}}$$

$$=\frac{r^{n-1}+r^{n-2}+\ldots+r+1}{ar^{n-1}}$$

Now,

L.H.S = P^2R^n

$$=\left[a^n\times r^{\frac{n(n-1)}{2}}\right]^2\left[\frac{r^{n-1}+r^{n-2}+\ldots+r+1}{ar^{n-1}}\right]^n$$

$$=a^{2n}r^{n(n-1)}\times\frac{\left(r^n-1\right)^n}{a^n r^{n(n-1)}\left(r-1\right)^n}$$

$$=\left[a^n\left(\frac{r^n-1}{r-1}\right)^n\right]$$

$$=\left[a\frac{\left(r^n-1\right)}{\left(r-1\right)}\right]^n$$

$= S^n$

= R.H.S

Hence, $P^2R^n = S^n$

15. The $p^{\text{th}}, q^{\text{th}}$ and r^{th} terms of an A.P. are a, b, c respectively. Show that $(q-r)a+(r-p)b+(p-q)c=0$

15. Let t and d be the first term and the common difference of the A.P. respectively.

The n^{th} term of an A.P. is given by, $a_n=t+(n-1)d$

Therefore,

$a_p=t+(p-1)d=a$ ....(1)

$a_q=t+(q-1)d=b$ ....(2)

$a_r=t+(r-1)d=c$ ....(3)

Subtracting equation (2) from (1), we obtain

$(p-1-q+1)d=a-b$

$\Rightarrow (p-q)d=a-b$

$\therefore d=\dfrac{a-b}{p-q}$

Subtracting equation (3) from (2), we obtain

$(q-1-r+1)d=b-c$

$\Rightarrow (q-r)d=b-c$

$\Rightarrow d=\dfrac{b-c}{q-r}$

Equating both the values of d obtained in (4) and (5), we obtain

$\dfrac{a-b}{p-q}=\dfrac{b-c}{q-r}$

$\Rightarrow (a-b)(q-r)=(b-c)(p-q)$

$\Rightarrow aq-bq-ar+br=bp-bq-cp+cq$

$\Rightarrow bp-cp+cq-aq+ar-br=0$

$\Rightarrow (-aq+ar)+(bp-br)+(-cp+cq)=0$ (By rearranging terms)

$\Rightarrow -a(q-r)-b(r-p)-c(p-q)=0$

$\Rightarrow a(q-r)+b(r-p)+c(p-q)=0$

16. If $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right),b\left(\dfrac{1}{c}+\dfrac{1}{a}\right),c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

16. Given that $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right),b\left(\dfrac{1}{c}+\dfrac{1}{a}\right),c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in AP.

If $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right),b\left(\dfrac{1}{c}+\dfrac{1}{a}\right),c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in AP

Adding 1 to each term

$a\left(\dfrac{1}{b}+\dfrac{1}{c}\right)+1,b\left(\dfrac{1}{c}+\dfrac{1}{a}\right)+1,c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)+1$ are in AP

$a\left(\dfrac{1}{b}+\dfrac{1}{c}\right)+\dfrac{a}{a},b\left(\dfrac{1}{c}+\dfrac{1}{a}\right)+\dfrac{b}{b},c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)+\dfrac{c}{c}$ are in AP

$a\left(\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{a}\right),b\left(\dfrac{1}{c}+\dfrac{1}{a}+\dfrac{1}{b}\right),c\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$ are in AP

Divide each term by $\left(\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{a}\right)$

$$\frac{a\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)}{\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)}, \frac{b\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)}{\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)}, \frac{c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \text{ are in AP}$$

Hence, a, b, c are in AP

Hence Proved.

17. If a,b,c,d are in G.P, prove that $\left(a^n+b^n\right),\left(b^n+c^n\right),\left(c^n+d^n\right)$ are in G.P.

17. It is given that a,b,c, and d are in G.P.

$: b^2 = ac$ ....(1)

$c^2 = bd$ ....(2)

$ad = bc$ ....(3)

It has to be proved that $\left(a^n+b^n\right),\left(b^n+c^n\right),\left(c^n+d^n\right)$ are in G.P. i.e.,

$$\left(b^n+c^n\right)^2=\left(a^n+b^n\right)\left(c^n+d^n\right)$$

Consider L.H.S.

$$\left(b^n+c^n\right)^2=b^{2n}+2b^nc^n+c^{2n}$$

$$=\left(b^2\right)^n+2b^nc^n+\left(c^2\right)^n$$

$$=(ac)^n+2b^nc^n+(bd)^n \text{ [Using } (1) \text{ and (2)}$$

$$=a^nc^n+b^nc^n+b^nc^n+b^nd^n$$

$$=a^nc^n+b^nc^n+a^nd^n+b^nd^n \text{ [Using (3)]}$$

$$=c^n\left(a^n+b^n\right)+d^n\left(a^n+b^n\right)$$

$$=\left(a^n+b^n\right)\left(c^n+d^n\right)= \text{R.H.S.}$$

$$\therefore \left(b^n+c^n\right)^2=\left(a^n+b^n\right)\left(c^n+d^n\right)$$

Thus, $\left(a^n+b^n\right)$, $\left(b^n+c^n\right)$, and $\left(c^n+d^n\right)$ are in G.P.

18. If a and b are the roots of $x^2 - 3x + p = 0$ and c, d are roots of $x^2 - 12x + q = 0$, where a, b, c, d form a G.P. Prove that $(q + p): (q - p) = 17:16$.

18. Given that a and b are roots of $x^2-3x + p=0$

$\therefore$ a + b = 3 and ab= p …(1)

[∵ <u>If α and β are roots of the equation $ax^2 + bx + c=0$ then $\alpha + \beta=-b/a$ and $\alpha\beta=c/a$</u>.]
It is given that c and d are roots of $x^2-12x + q=0$
∴ $c + d = 12$ and $cd=q$ …(2)
[∵ <u>If α and β are roots of the equation $ax^2 + bx + c=0$ then $\alpha + \beta=-b/a$ and $\alpha\beta=c/a$</u>.]
Also given that a, b, c, d are in G.P.
Let a, b, c, d be the first four terms of a G.P.
So, a = a
b = ar
c = ar^2
d = ar^3
Now,

$$\text{L.H.S} = \frac{q+p}{q-p}$$

$$= \frac{cd+ab}{cd-ab}$$

$$= \frac{ar^2ar^3+a^2r}{ar^2ar^3-a^2r}$$

$$= \frac{a^2r\left(r^4+1\right)}{a^2r\left(r^4-1\right)}$$

$$= \frac{\left(r^4+1\right)}{\left(r^4-1\right)}$$

Now, From (1)
a + b=3
⇒ a + ar=3
⇒ a(1 + r)=3 ...(3)
From (2),
c + d=12
⇒ $ar^2 + ar^3=12$
⇒ $ar^2(1 + r)=12$ ...(4)
Dividing equation (4) by (3), we get -
$r^2 = 4$
∴ $r^4 = 16$
putting the value of r^4 in L.H.S, we get -

$$\frac{q+p}{q-p} = \frac{16+1}{16-1} = \frac{17}{16}$$

Hence proved.

19. The ratio of the A.M and G.M. of two positive numbers a and b, is $m : n$. Show that $a : b = \left(m+\sqrt{m^2-n^2}\right) : \left(m-\sqrt{m^2-n^2}\right)$

19. Let the two numbers be a and b.

A.M $= \dfrac{a+b}{2}$ and G.M. $= \sqrt{ab}$

According to the given condition,

$$\frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$$

$$\Rightarrow \frac{(a+b)^2}{4(ab)} = \frac{m^2}{n^2}$$

$$\Rightarrow (a+b)^2 = \frac{4abm^2}{n^2}$$

$$\Rightarrow (a+b) = \frac{2\sqrt{ab}m}{n}$$

Using this in the identity $(a-b)^2 = (a+b)^2 - 4ab$, we obtain

$$(a-b)^2 = \frac{4abm^2}{n^2} - 4ab = \frac{4ab\left(m^2-n^2\right)}{n^2}$$

$$\Rightarrow (a-b) = \frac{2\sqrt{ab}\sqrt{m^2-n^2}}{n}$$

Adding (1) and (2), we obtain

20. If a, b, c are in A.P.; b, c, d are in G.P. and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. prove that a, c, e are in G.P.

20. It is given that a, b, c are in AP

∴ b = (a + c)/2 …(1)

Also given that b, c, d are in GP

∴ $c^2 = bd$ …(2)

Also,

$\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in AP

So, their common difference is same

$$\frac{1}{d}-\frac{1}{c}=\frac{1}{e}-\frac{1}{d}$$

$$\Rightarrow \frac{1}{d}+\frac{1}{d}=\frac{1}{c}+\frac{1}{e}$$

$$\Rightarrow \frac{2}{d}=\frac{1}{c}+\frac{1}{e}$$

$$\Rightarrow \frac{2}{d}=\frac{c+e}{ce}$$

$$\Rightarrow \frac{d}{2}=\frac{ce}{c+e}$$

$$\therefore d=\frac{2ce}{c+e} \qquad \ldots(3)$$

We need to show that a, c, e are in GP

i.e $c^2 = ae$

From (2), we have

$c^2 = bd$

Putting value of $b=\frac{a+c}{2}$ & $d=\frac{2ce}{c+e}$

$$c^2=\left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right)$$

$$\Rightarrow c^2=\frac{(a+c)(ce)}{(c+e)}$$

$$\Rightarrow \frac{c^2}{c}=\frac{e(a+c)}{(c+e)}$$

$$\Rightarrow c=\frac{e(a+c)}{(c+e)}$$

$\Rightarrow c(c + e) = e(a + c)$

$\Rightarrow c^2 + ce = ea + ec$

$\Rightarrow c^2 = ea$

Thus, a, c, e are in GP.

Hence, Proved.

21. Find the sum of the following series up to n terms:

(i) $5+55+555+\dots$

(ii) $.6+.66+.666+\dots$

21. (i) $5+55+555+\dots$

Let $S_n = 5+55+555+\dots$. to n terms

$$=\frac{5}{9}[9+99+999+\dots \text{ to n terms}]$$

$$=\frac{5}{9}\left[(10-1)+(10^2-1)+(10^3-1)+\dots \text{ to n terms}\right]$$

$$=\frac{5}{9}\left[(10+10^2+10^3+\dots\text{n terms})-(1+1+\dots\text{nterms})\right]$$

$$=\frac{5}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$$

$$=\frac{5}{9}\left[\frac{10(10^n-1)}{9}-n\right]$$

$$=\frac{50}{81}(10^n-1)-\frac{5n}{9}$$

(ii) $.6+.66+.666+\dots$

Let $S_n = 06.+0.66+0.666+\dots$ to n terms

$$=6[0.1+0.11+0.111+\dots \text{ to } n \text{ terms}]$$

$$=\frac{6}{9}[0.9+0.99+0.999+\dots \text{ to } n \text{ terms}]$$

$$=\frac{6}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\dots \text{ to } n \text{ terms}\right]$$

$$=\frac{2}{3}\left[(1+1+\dots\text{n terms})-\frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\dots n \text{ terms}\right)\right]$$

$$= \frac{2}{3}\left[n - \frac{1}{10}\left(\frac{1-\left(\frac{1}{10}\right)^{n}}{1-\frac{1}{10}} \right) \right]$$

$$= \frac{2}{3}n - \frac{2}{30} \times \frac{10}{9}\left(1-10^{-n}\right)$$

$$= \frac{2}{3}n - \frac{2}{27}\left(1-10^{-n}\right)$$

22. Find the 20^{th} term of the series 2 × 4 + 4 × 6 + 6 × 8 + ... + n terms.

22. The given series is in the form of multiplication of two different APs.

So, the nth term of given series is equal to the multiplication of their nth term.

The First AP is given as follows: -

2, 4, 6…

where, first term(a) = 2

common difference(d) = 4 - 2 = 2

∴ n^{th} term = a + (n - 1)d

= 2 + (n - 1)2

= 2 + 2n - 2

= 2n

The Second AP is given as follows: -

4, 6, 8…

where, first term(a) = 4

common difference(d) = 6 - 4 = 2

∴ n^{th} term = a + (n - 1)d

= 4 + (n - 1)2

= 4 + 2n - 2

= 2n + 2

Now,

a_n = [n^{th} term of 2, 4, 6…] × [n^{th} term of 4, 6, 8…]

= (2n) × (2n + 2)

= $4n^2 + 4n$

Thus, the n^{th} term of series 2 × 4 + 4 × 6 + 6 × 8 + ... is

$a_n = 4n^2 + 4n$

∴ $a_{20} = 4 \times (20)^2 + 4 \times 20 = 1600 + 80 = 1680$

Hence, 20^{th} term of series is 1680.

23. Find the sum of the first n terms of the series: $3+7+13+21+31+\ldots$

23. The given series is $3+7+13+21+31+\ldots$

$S=3+7+13+21+31+\ldots+a_{n-1}+a_n$

$S=3+7+13+21+\ldots+a_{n-2}+a_{n-1}+a_n$

On subtracting both the equations, we obtain

$S-S=\left[3+\left(7+13+21+31+\ldots+a_{n-1}+a_n\right)\right]-[(3+7+13+21+$

$31+\ldots+a_{n-1})$

$+a_n]$

$\mathrm{S-S}=3+\left[(7-3)+(13-7)+(21-13)+\ldots+\left(a_n-a_{n-1}\right)\right]-a_n$

$0=3+[4+6+8+\ldots(n-1)\text{ terms }]-a_n$

$a_n=3+\left[4+6+8+\ldots(n-1)\text{ terms}\right]$

$\Rightarrow \mathrm{a_n}=3+\left(\frac{\mathrm{n}-1}{2}\right)\left[2\times4+(\mathrm{n}-1-1)2\right]$

$=3+\left(\frac{\mathrm{n}-1}{2}\right)\left[8+(\mathrm{n}-2)2\right]$

$=3+\frac{(n-1)}{2}(2n+4)$

$=3+(n-1)(n+2)$

$=3+\left(n^2+n-2\right)$

$=n^2+n+1$

$\therefore \sum_{k=1}^{n}a_k=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$

$=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}+n$

$=n\left[\frac{(n+1)(2n+1)+3(n+1)+6}{6}\right]$

$=n\left[\frac{2n^2+3n+1+3n+3+6}{6}\right]$

$$= n\left[\frac{2n^2+6n+10}{6}\right]$$

$$= \frac{n}{3}\left(n^2+3n+5\right)$$

24. If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3(1 + 8S_1)$.

24. According to question -

$S_1 = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

$S_2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$S_3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Now,

R.H.S = $S_3(1 + 8S_1)$

$$= \left(\frac{n(n+1)}{2}\right)^2\left(1+8\frac{n(n+1)}{2}\right)$$

$$= \frac{n^2(n+1)^2}{4}\times\left[1+4n(n+1)\right]$$

$$= \frac{n^2(n+1)^2}{4}\times\left[4n^2+4n+1\right]$$

$$= \frac{n^2(n+1)^2}{4}\times(2n+1)^2$$

$$= \left(\frac{n(n+1)(2n+1)}{2}\right)^2$$

$$= \left(3\times\frac{n(n+1)(2n+1)}{6}\right)^2$$

$$= 9\left(\frac{n(n+1)(2n+1)}{6}\right)^2$$

$$= 9S_2^2$$

= R.H.S

Hence, L.H.S = R.H.S

Hence Proved.

25. Find the sum of the following series up to n terms: $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$

25. The n^{th} term of the given series is $\frac{1^3+2^3+3^3+\ldots+n^3}{1+3+5+\ldots+(2n-1)}=\frac{\left[\frac{n(n+1)}{2}\right]^2}{1+3+5+\ldots+(2n-1)}$

Here, 1,3,5, $(2n-1)$ is an A.P. with first term a, last term $(2n-1)$ and number of terms

as n $\therefore 1+3+5+\ldots.+(2n-1)=\frac{n}{2}\left[2\times1+(n-1)2\right]=n^2$

$$\therefore a_n=\frac{n^2(n+1)^2}{4n^2}=\frac{(n+1)^2}{4}=\frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{4}$$

$$\therefore S_n=\Sigma_{K=1}^{n}a_K=\Sigma_{K=1}^{n}\left(\frac{1}{4}K^2+\frac{1}{2}K+\frac{1}{4}\right)$$

$$=\frac{1}{4}\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\frac{n(n+1)}{2}+\frac{1}{4}n$$

$$=\frac{n\left[(n+1)(2n+1)+6(n+1)+6\right]}{24}$$

$$=\frac{n\left[2n^2+3n+1+6n+6+6\right]}{24}$$

$$=\frac{n\left(2n^2+9n+13\right)}{24}$$

26. Show that

$$\frac{1\times2^2+2\times3^2+\ldots n\times(n+1)^2}{1^2\times2+2^2\times3+\ldots n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

26. Taking L.H.S

$$\frac{1\times 2^2+2\times 3^2+\ldots n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots n^2\times(n+1)}$$

first we will solve the numerator & denominator separately

Let numerator be

$S_1 = 1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \ldots + n \times (n + 1)^2$

n^{th} term is $n \times (n + 1)^2$

Let $a_n = n(n + 1)^2$

$= n(n^2 + 2n + 1)$

$= n^3 + 2n^2 + n$

Now, $S_1 = \sum_{n=1}^{n} a_n$

$$= \sum_{n=1}^{n} n^3 + 2n^2 + n$$

$$= \sum_{n=1}^{n} n^3 + 2\sum_{n=1}^{n} n^2 + \sum_{n=1}^{n} n$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + 2\times\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n(n+1)+4(2n+1)+6}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+8n+4+6}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+11n+10}{6}\right]$$

$$= \frac{n(n+1)}{12}(3n^2+11n+10)$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12}$$

Let denominator be

$S_2 = 1^2 \times 2 + 2^2 \times 3 + \ldots + n^2 \times (n + 1)$

n^{th} term is $n^2 \times (n + 1)$

Let $b_n = n^2(n + 1) = n^3 + n^2$

Now,

$S_2 = \sum_{n=1}^{n} b_n$

$= \sum_{n=1}^{n} n^3 + n^2$

$= \sum_{n=1}^{n} n^3 + \sum_{n=1}^{n} n^2$

$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}$

$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{(2n+1)}{3}\right]$

$= \frac{n(n+1)}{2}\left[\frac{3n(n+1)+2(2n+1)}{6}\right]$

$= \frac{n(n+1)}{12}\left[3n^2 + 7n + 2\right]$

$= \frac{n(n+1)(n+2)(3n+1)}{12}$

Now,

L.H.S $= \frac{1\times 2^2 + 2\times 3^2 + \ldots n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + \ldots n^2\times(n+1)}$

$= \frac{S_1}{S_2}$

$= \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}$

$= \frac{3n+5}{3n+1}$

= R.H.S

Hence, L.H.S = R.H.S

Hence Proved.

27. A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual installments of Rs 500 plus 12% interest on the unpaid amount. How much will be the tractor cost him?

27. It is given that the farmer pays Rs 6000 in cash.

Therefore, unpaid amount = Rs 12000 − Rs 6000 = Rs 6000

According to the given condition, the interest paid annually is 12% of $6000, 12\%$ of $5500, 12\%$ of $5000 \ldots 12\%$ of 500

Thus, total interest to be paid

$= 12\% \text{ of} 6000 + 12\% \text{ of} 5500 + 12\% \text{ of} 5000 + \ldots + 12\% \text{ of} 500$

$= 12\% \text{ of} (6000 + 5500 + 5000 + \ldots + 500)$

$= 12\% \text{ of} (500 + 1000 + 1500 + \ldots + 6000)$

Now, the series $500, 1000, 1500 \ldots 6000$ is an A.P. with both the first term and common difference equal to 500.

Let the number of terms of the A.P. be n.

$\therefore 6000 = 500 + (n-1)500$

$\Rightarrow 1 + (n-1) = 12$

$\Rightarrow n = 12$

$\therefore$ Sum of the A.P

$= \frac{12}{2}\left[2(500) + (12-1)(500)\right] = 6\left[1000 + 5500\right] = 6(6500) = 39000$

Thus, total interest to be paid

$= 12\% \text{ of} (500 + 1000 + 1500 + \ldots + 6000)$

$= 12\% \text{ of} 39000 = \text{ Rs} 4680$

Thus, cost of tractor $= ($ Rs $12000 +$ Rs $4680) =$ Rs 16680

28. Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual instalment of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

28. Amount Paid to buy scooter = Rs. 22,000

Shamshad Pays Cash = Rs. 4000

Remaining Balance = Rs. (22000 - 4000) = 18000

Annual Instalment = Rs 1000 + interest@10% on unpaid amount

1st Instalment

Unpaid Amount = Rs. 18000

Interest on Unpaid Amount = (10/100) × 18000 = 1800

Amount of Instalment = Rs. 1000 + Rs. 1800 = Rs. 2800

2nd Instalment

Unpaid Amount = Rs. (18000 - 1000) = Rs. 17000

Interest on Unpaid Amount = (10/100) × 17000 = 1700
Amount of Instalment = Rs. 1000 + Rs. 1700 = Rs. 2700
3rd Instalment
Unpaid Amount = Rs. (17000 - 1000) = Rs. 16000
Interest on Unpaid Amount = (10/100) × 16000 = 1600
Amount of Instalment = Rs. 1000 + Rs. 1600 = Rs. 2600
Total no. of Instalments = 18000/1000 = 18
Thus, Annual Instalments are 2800, 2700, 2600, …upto 18 terms
Since the common difference between the consecutive terms is constant. Thus, Annual Instalments are in AP.
Here
first term(a) = 2800
Common difference(d) = 2700 - 2800 = - 100
Number of terms(n) = 18
Total amount paid in 12 instalments is given by -
$S_n = (n/2)[2a + (n - 1)d]$
$\therefore S_{18} = (18/2)[2(2800) + (18 - 1)(- 100)]$
$= 9[5600 + 17(- 100)]$
$= 9[5600 - 1700]$
$= 9 \times 3900$
$= 35100$
Hence, total amount paid in 12 Instalments = Rs 35100
Hence,
Total Cost of Tractor
= Amount paid earlier + Amount paid in 12 Instalments
= Rs. (4000 + 35100)
= Rs. 39100

29. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8^{th} set of letter is mailed.

29. The numbers of letters mailed forms a G.P.: $4, 4^2, \ldots 4^8$
First term $= 4$
Common ratio $= 4$
Number of terms $= 8$
It is known that the sum of n terms of a G.P. is given by

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$\therefore S_8 = \frac{4(4^8-1)}{4-1} = \frac{4(65536-1)}{3} = \frac{4(65535)}{3} = 4(21845) = 87380$$

It is given that the cost to mail one letter is 50 paisa.

$\therefore$ Cost of mailing 87380 letters = Rs $87380 \times \frac{50}{100}$ = Rs 43690

Thus, the amount spent when 8^{th} set of letter is mailed is Rs 43690 .

30. A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15^{th} year since he deposited the amount and also calculate the total amount after 20 years.

30. In simple interest, the interest remains same in all year.

Interest per year = 10000 × 5% = 500

Hence,

Amount in 1st year = Rs. 10000

Amount in 2nd year = Amount in 1st year + Interest

= 10000 + 500

= 10500

Amount in 3rd year = Amount in 2nd year + Interest

= 10500 + 500

= 11000

Hence, the series becomes

10000, 10500, 11000,…

Since the common difference between the consecutive terms is constant. Thus, Above Series are in AP.

Where,

first term(a) = 10000

common difference(d) = 10500 - 10000 = 500

Amount in 15th year is given by putting n = 15 in

$a_n = a + (n - 1)d$

$\therefore a_{15} = 10000 + (15 - 1)500$

= 10000 + 14(500)

= 10000 + 7000

= 17000

Thus, amount in 15th year is Rs. 17000.

Also, Amount after 20 years

$= a_{21}$

= 10000 + (21 - 1)500

= 10000 + 20 × 500

= 10000 + 10000

= 20000

Thus, amount after 20th year is Rs. 20000.

31. A manufacturer reckons that the value of a machine, which costs him Rs 15625 , will depreciate each year by 20% . Find the estimated value at the end of 5 years.

31. Cost of machine = Rs 15625

Machine depreciates by 20% every year.

Therefore, its value after every year is 80% of the original cost i.e., $\frac{4}{5}$ of the original cost.

$\therefore$ Value at the end of 5 years $= 15625 \times \underbrace{\frac{4}{5} \times \frac{4}{5} \times \ldots \times \frac{4}{5}}_{5\text{times}} = 5 \times 1024 = 5120$

Thus, the value of the machine at the end of 5 years is Rs 5120.

32. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

32. Let total work = 1

and let total work be completed in 'n' days

Work done in 1 day = 1/n

This is the work done by 150 workers

Work done by 1 worker in one day = 1/150n

<u>Case 1</u>: -

No. of workers = 150

Work done per worker in 1 day = 1/150n

Total work done in 1 day = 150/150n

<u>Case 2</u>: -

No. of workers = 146

Work done per worker in 1 day = 1/150n

Total work done in 1 day = 146/150n

<u>Case 3</u>: -

No. of workers = 142

Work done per worker in 1 day = 1/150n

Total work done in 1 day = 142/150n

Given that

In this manner it took 8 more days to finish the work i.e. work finished in (n + 8) days.

$$\therefore \frac{150}{150n} + \frac{146}{150n} + \frac{142}{150n} + \ldots + to(n+8)\,terms = 1$$

$$\Rightarrow \frac{1}{150n}\left[150 + 146 + 142 + \ldots + \; to(n+8)\,terms\right] = 1$$

$$\Rightarrow 150 + 146 + 142 + \ldots + \; to(n+8)\,terms = 150n \qquad \ldots(1)$$

Now,

$150 + 146 + 142 + \ldots + \; to(n+8)\,terms$ is an AP

where,

first term(a) = 150

common difference(d) = 146 - 150 = - 4

we know that

Sum of n terms of AP(S_n) = (n/2)[2a + (n - 1)d]

putting n = n + 8, a = 150 & d = - 4

S_{n+8} = [(n + 8)/2] × [2(150) + (n + 8 - 1)(- 4)]

= [(n + 8)/2] × [300 + (n + 7)(- 4)]

= [(n + 8)/2] × [300 - 4n - 28]

= [(n + 8)/2] × [272 - 4n]

= (n + 8) × (136 - 2n)

= - $2n^2$ + 120n + 1088

From (1),

S_{n+8} = 150n

⇒ - $2n^2$ + 120n + 1088 = 150n

⇒ - $2n^2$ - 30n + 1088 = 0

⇒ - 2(n^2 + 15n - 544) = 0

⇒ (n^2 + 15n - 544) = 0

⇒ n^2 + 32n - 17n - 544 = 0

⇒ n(n + 32) - 17(n + 32) = 0

⇒ (n - 17)(n + 32) = 0

∴ n = 17

because n = - 32 is invalid as no. of days can't be - ve.

Hence, n =17

Thus, the work was completed in n + 8 days i.e. 17 + 8 = 25 days