

EXERCISE 1.3

1. Show that the square of any positive integer is either of the form $4q$ or $4q + 1$ for some integer q .

Solution:

According to Euclid's division lemma,

$$a = bq + r$$

According to the question,

When $b = 4$,

$$a = 4k + r, 0 \leq r < 4$$

When $r = 0$, we get, $a = 4k$

$$a^2 = 16k^2 = 4(4k^2) = 4q, \text{ where } q = 4k^2$$

When $r = 1$, we get, $a = 4k + 1$

$$a^2 = (4k + 1)^2 = 16k^2 + 1 + 8k = 4(4k^2 + 2) + 1 = 4q + 1, \text{ where } q = k(4k + 2)$$

When $r = 2$, we get, $a = 4k + 2$

$$a^2 = (4k + 2)^2 = 16k^2 + 4 + 16k = 4(4k^2 + 4k + 1) = 4q, \text{ where } q = 4k^2 + 4k + 1$$

When $r = 3$, we get, $a = 4k + 3$

$$a^2 = (4k + 3)^2 = 16k^2 + 9 + 24k = 4(4k^2 + 6k + 2) + 1 \\ = 4q + 1, \text{ where } q = 4k^2 + 6k + 2$$

Therefore, the square of any positive integer is either of the form $4q$ or $4q + 1$ for some integer q .

Hence Proved.

2. Show that cube of any positive integer is of the form $4m$, $4m + 1$ or $4m + 3$, for some integer m .

Solution:

Let a be any positive integer and $b = 4$.

According to Euclid Division Lemma,

$$a = bq + r \quad [0 \leq r < b]$$

$$a = 3q + r \quad [0 \leq r < 4]$$

According to the question, the possible values of r are,

$$r = 0, r = 1, r = 2, r = 3$$

When $r = 0$,

$$a = 4q + 0$$

$$a = 4q$$

Taking cubes on LHS and RHS,

We have,

$$a^3 = (4q)^3$$

$$a^3 = 4(16q^3)$$

$$a^3 = 9m \quad [\text{where } m \text{ is an integer} = 16q^3]$$

When $r = 1$,

$$a = 4q + 1$$

Taking cubes on LHS and RHS,

We have,

$$a^3 = (4q + 1)^3$$

$$a^3 = 64q^3 + 1^3 + 3 \times 4q \times 1(4q + 1)$$

$$a^3 = 64q^3 + 1 + 48q^2 + 12q$$

$$a^3 = 4(16q^3 + 12q^2 + 3q) + 1$$

$$a^3 = 4m + 1 \quad [\text{where } m \text{ is an integer} = 16q^3 + 12q^2 + 3q]$$

When $r = 2$,

$$a = 4q + 2$$

Taking cubes on LHS and RHS,

We have,

$$a^3 = (4q + 2)^3$$

$$a^3 = 64q^3 + 2^3 + 3 \times 4q \times 2(4q + 2)$$

$$a^3 = 64q^3 + 8 + 96q^2 + 48q$$

$$a^3 = 4(16q^3 + 2 + 24q^2 + 12q)$$

$$a^3 = 4m \quad [\text{where } m \text{ is an integer} = 16q^3 + 2 + 24q^2 + 12q]$$

When $r = 3$,

$$a = 4q + 3$$

Taking cubes on LHS and RHS,

We have,

$$a^3 = (4q + 3)^3$$

$$a^3 = 64q^3 + 27 + 3 \times 4q \times 3(4q + 3)$$

$$a^3 = 64q^3 + 24 + 3 + 144q^2 + 108q$$

$$a^3 = 4(16q^3 + 36q^2 + 27q + 6) + 3$$

$$a^3 = 4m + 3 \quad [\text{where } m \text{ is an integer} = 16q^3 + 36q^2 + 27q + 6]$$

Hence, the cube of any positive integer is in the form of $4m$, $4m+1$ or $4m+3$.

3. Show that the square of any positive integer cannot be of the form $5q + 2$ or $5q + 3$ for any integer q .

Solution:

Let the positive integer = a

According to Euclid's division lemma,

$$a = 5m + r$$

According to the question, $b = 5$

$$a = 5m + r$$

So, $r = 0, 1, 2, 3, 4$

When $r = 0$, $a = 5m$.

When $r = 1$, $a = 5m + 1$.

When $r = 2$, $a = 5m + 2$.

When $r = 3$, $a = 5m + 3$.

When $r = 4$, $a = 5m + 4$.

Now,

When $a = 5m$

$$a^2 = (5m)^2 = 25m^2$$

$$a^2 = 5(5m^2) = 5q, \text{ where } q = 5m^2$$

When $a = 5m + 1$

$$a^2 = (5m + 1)^2 = 25m^2 + 10m + 1$$

$$a^2 = 5(5m^2 + 2m) + 1 = 5q + 1, \text{ where } q = 5m^2 + 2m$$

When $a = 5m + 2$

$$a^2 = (5m + 2)^2$$

$$a^2 = 25m^2 + 20m + 4$$

$$a^2 = 5(5m^2 + 4m) + 4$$

$$a^2 = 5q + 4 \text{ where } q = 5m^2 + 4m$$

$$\text{When } a = 5m + 3$$

$$a^2 = (5m + 3)^2 = 25m^2 + 30m + 9$$

$$a^2 = 5(5m^2 + 6m + 1) + 4$$

$$a^2 = 5q + 4 \text{ where } q = 5m^2 + 6m + 1$$

$$\text{When } a = 5m + 4$$

$$a^2 = (5m + 4)^2 = 25m^2 + 40m + 16$$

$$a^2 = 5(5m^2 + 8m + 3) + 1$$

$$a^2 = 5q + 1 \text{ where } q = 5m^2 + 8m + 3$$

Therefore, square of any positive integer cannot be of the form $5q + 2$ or $5q + 3$.

Hence Proved.

4. Show that the square of any positive integer cannot be of the form $6m + 2$ or $6m + 5$ for any integer m .

Solution:

Let the positive integer = a

According to Euclid's division algorithm,

$$a = 6q + r, \text{ where } 0 \leq r < 6$$

$$a^2 = (6q + r)^2 = 36q^2 + r^2 + 12qr \quad [\because (a+b)^2 = a^2 + 2ab + b^2]$$

$$a^2 = 6(6q^2 + 2qr) + r^2 \quad \dots(i), \text{ where, } 0 \leq r < 6$$

When $r = 0$, substituting $r = 0$ in Eq.(i), we get

$$a^2 = 6(6q^2) = 6m, \quad \text{where, } m = 6q^2 \text{ is an integer.}$$

When $r = 1$, substituting $r = 1$ in Eq.(i), we get

$$a^2 + 6(6q^2 + 2q) + 1 = 6m + 1, \quad \text{where, } m = (6q^2 + 2q) \text{ is an integer.}$$

When $r = 2$, substituting $r = 2$ in Eq.(i), we get

$$a^2 = 6(6q^2 + 4q) + 4 = 6m + 4, \quad \text{where, } m = (6q^2 + 4q) \text{ is an integer.}$$

When $r = 3$, substituting $r = 3$ in Eq.(i), we get

$$a^2 = 6(6q^2 + 6q) + 9 = 6(6q^2 + 6q) + 6 + 3$$

$$a^2 = 6(6q^2 + 6q + 1) + 3 = 6m + 3, \quad \text{where, } m = (6q^2 + 6q + 1) \text{ is integer.}$$

When $r = 4$, substituting $r = 4$ in Eq.(i) we get

$$a^2 = 6(6q^2 + 8q) + 16$$

$$= 6(6q^2 + 8q) + 12 + 4$$

$$\Rightarrow a^2 = 6(6q^2 + 8q + 2) + 4 = 6m + 4, \text{ where, } m = (6q^2 + 8q + 2) \text{ is integer.}$$

When $r = 5$, substituting $r = 5$ in Eq.(i), we get

$$a^2 = 6(6q^2 + 10q) + 25 = 6(6q^2 + 10q) + 24 + 1$$

$$a^2 = 6(6q^2 + 10q + 4) + 1 = 6m + 1, \text{ where, } m = (6q^2 + 10q + 4) \text{ is integer.}$$

Hence, the square of any positive integer cannot be of the form $6m + 2$ or $6m + 5$ for any integer m .

Hence Proved

5. Show that the square of any odd integer is of the form $4q + 1$, for some integer q .

Solution:

Let a be any odd integer and $b = 2$.

According to Euclid's algorithm,

$$a = 2m + r \text{ for some integer } m \geq 0$$

And $r = 0, 1, 2, 3$ because $0 \leq r < 4$.

So, we have that,

$a = 4m$ or $4m + 1$ or $4m + 2$ or $4m + 3$ So, $a = 4m + 1$ or $4m + 3$

We know that, a cannot be $4m$ or $4m + 2$, as they are divisible by 2.

$$(4m + 1)^2 = 16m^2 + 8m + 1$$

$$= 4(4m^2 + 2m) + 1$$

$$= 4q + 1, \text{ where } q \text{ is some integer and } q = 4m^2 + 2m.$$

$$(4m + 3)^2 = 16m^2 + 24m + 9$$

$$= 4(4m^2 + 6m + 2) + 1$$

$$= 4q + 1, \text{ where } q \text{ is some integer and } q = 4m^2 + 6m + 2$$

Therefore, Square of any odd integer is of the form $4q + 1$, for some integer q .

Hence Proved.

6. If n is an odd integer, then show that $n^2 - 1$ is divisible by 8.

Solution:

We know that any odd positive integer n can be written in form $4q + 1$ or $4q + 3$.

So, according to the question,

When $n = 4q + 1$,

Then $n^2 - 1 = (4q + 1)^2 - 1 = 16q^2 + 8q + 1 - 1 = 8q(2q + 1)$, is divisible by 8.

When $n = 4q + 3$,

Then $n^2 - 1 = (4q + 3)^2 - 1 = 16q^2 + 24q + 9 - 1 = 8(2q^2 + 3q + 1)$, is divisible by 8.

So, from the above equations, it is clear that, if n is an odd positive integer $n^2 - 1$ is divisible by 8.

Hence Proved.

7. Prove that if x and y are both odd positive integers, then $x^2 + y^2$ is even but not divisible by 4.

Solution:

Let the two odd positive numbers x and y be $2k + 1$ and $2p + 1$, respectively

$$\begin{aligned} \text{i.e., } x^2 + y^2 &= (2k + 1)^2 + (2p + 1)^2 \\ &= 4k^2 + 4k + 1 + 4p^2 + 4p + 1 \\ &= 4k^2 + 4p^2 + 4k + 4p + 2 \\ &= 4(k^2 + p^2 + k + p) + 2 \end{aligned}$$

Thus, the sum of square is even the number is not divisible by 4

Therefore, if x and y are odd positive integer, then $x^2 + y^2$ is even but not divisible by four.

Hence Proved