EXERCISE 20.5

If a, b, c are in G.P., prove that log a, log b, log c are in A.P. Solution:

It is given that a, b and c are in G.P.

$b^2 = ac$ {using property of geometric mean}

Now, apply log on both the sides we get,

$\log b^2 = \log (ac)$

$\log (b)^2 = \log a + \log c$

$2 \log b = \log a + \log c$

∴ log a, log b, log c are in A.P

1. If a, b, c are in G.P., prove that $1/\log_a m$, $1/\log_b m$, $1/\log_c m$ are in A.P.

Solution:

Given:

a, b and c are in GP

$b^2 = ac$ {property of geometric mean}

Apply log on both sides with base m

$\log_m b^2 = \log_m ac$

$\log_m b^2 = \log_m a + \log_m c$ {using property of log}

$2\log_m b = \log_m a + \log_m c$

$2/\log_b m = 1/\log_a m + 1/\log_c m$

∴ $1/\log_a m$, $1/\log_b m$, $1/\log_c m$ are in A.P.

2. Find k such that k + 9, k – 6 and 4 form three consecutive terms of a G.P.

Solution:

Let $a = k + 9$; $b = k-6$; and $c = 4$;

We know that a, b and c are in GP, then

$b^2 = ac$ {using property of geometric mean}

$(k - 6)^2 = 4(k + 9)$

$k^2 - 12k + 36 = 4k + 36$

$k^2 - 16k = 0$

$k = 0$ or $k = 16$

3. Three numbers are in A.P., and their sum is 15. If 1, 3, 9 be added to them respectively, they from a G.P. find the numbers.

Solution:

Let the first term of an A.P. be ‘a’ and its common difference be‘d’.
$a_1 + a_2 + a_3 = 15$
Where, the three number are: a, a + d, and a + 2d
So,
$a + a + d + a + 2d = 15$
$3a + 3d = 15$ or $a + d = 5$
$d = 5 – a$ … (i)

Now, according to the question:
a + 1, a + d + 3, and a + 2d + 9
they are in GP, that is:
$(a+d+3)/(a+1) = (a+2d+9)/(a+d+3)$
$(a + d + 3)^2 = (a + 2d + 9) (a + 1)$
$a^2 + d^2 + 9 + 2ad + 6d + 6a = a^2 + a + 2da + 2d + 9a + 9$
$(5 – a)^2 – 4a + 4(5 – a) = 0$
$25 + a^2 – 10a – 4a + 20 – 4a = 0$
$a^2 – 18a + 45 = 0$
$a^2 – 15a – 3a + 45 = 0$
$a(a – 15) – 3(a – 15) = 0$
$a = 3$ or $a = 15$
$d = 5 – a$
$d = 5 – 3$ or $d = 5 – 15$
$d = 2$ or $– 10$
Then,
For a = 3 and d = 2, the A.P is 3, 5, 7
For a = 15 and d = -10, the A.P is 15, 5, -5
∴ The numbers are 3, 5, 7 or 15, 5, – 5

4. The sum of three numbers which are consecutive terms of an A.P. is 21. If the second number is reduced by 1 and the third is increased by 1, we obtain three consecutive terms of a G.P. Find the numbers.
Solution:
Let the first term of an A.P. be ‘a’ and its common difference be‘d’.
$a_1 + a_2 + a_3 = 21$
Where, the three number are: a, a + d, and a + 2d
So,
$3a + 3d = 21$ or
$a + d = 7$.
$d = 7 – a$ …. (i)

Now, according to the question:
a, a + d – 1, and a + 2d + 1
they are now in GP, that is:
(a+d-1)/a = (a+2d+1)/(a+d-1)
$(a + d - 1)^2 = a(a + 2d + 1)$
$a^2 + d^2 + 1 + 2ad - 2d - 2a = a^2 + a + 2da$
$(7 - a)^2 - 3a + 1 - 2(7 - a) = 0$
$49 + a^2 - 14a - 3a + 1 - 14 + 2a = 0$
$a^2 - 15a + 36 = 0$
$a^2 - 12a - 3a + 36 = 0$
a(a – 12) – 3(a – 12) = 0
a = 3 or a = 12
d = 7 – a
d = 7 – 3 or d = 7 – 12
d = 4 or – 5
Then,
For a = 3 and d = 4, the A.P is 3, 7, 11
For a = 12 and d = -5, the A.P is 12, 7, 2
∴ The numbers are 3, 7, 11 or 12, 7, 2

5. The sum of three numbers a, b, c in A.P. is 18. If a and b are each increased by 4 and c is increased by 36, the new numbers form a G.P. Find a, b, c.
Solution:
Let the first term of an A.P. be 'a' and its common difference be'd'.
b = a + d; c = a + 2d.
Given:
a + b + c = 18
3a + 3d = 18 or a + d = 6.
d = 6 – a … (i)

Now, according to the question:
a + 4, a + d + 4, and a + 2d + 36
they are now in GP, that is:
(a+d+4)/(a+4) = (a+2d+36)/(a+d+4)
$(a + d + 4)^2 = (a + 2d + 36)(a + 4)$
$a^2 + d^2 + 16 + 8a + 2ad + 8d = a^2 + 4a + 2da + 36a + 144 + 8d$
$d^2 - 32a - 128 = 0$
$(6 - a)^2 - 32a - 128 = 0$

$36 + a^2 – 12a – 32a – 128 = 0$
$a^2 – 44a – 92 = 0$
$a^2 – 46a + 2a – 92 = 0$
$a(a – 46) + 2(a – 46) = 0$
$a = – 2$ or $a = 46$
$d = 6 – a$
$d = 6 – (– 2)$ or $d = 6 – 46$
$d = 8$ or $– 40$
Then,
For a = -2 and d = 8, the A.P is -2, 6, 14
For a = 46 and d = -40, the A.P is 46, 6, -34
∴ The numbers are – 2, 6, 14 or 46, 6, – 34

6. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an A.P. Find the numbers.
Solution:
Let the three numbers be a, ar, ar^2
According to the question
$a + ar + ar^2 = 56$ … (1)
Let us subtract 1,7,21 we get,
$(a – 1), (ar – 7), (ar^2 – 21)$
The above numbers are in AP.
If three numbers are in AP, by the idea of the arithmetic mean, we can write 2b = a + c
$2 (ar - 7) = a – 1 + ar^2 – 21$
$= (ar^2 + a) – 22$
$2ar – 14 = (56 - ar) - 22$
$2ar – 14 = 34 – ar$
$3ar = 48$
$ar = 48/3$
$ar = 16$
$a = 16/r$ …. (2)
Now, substitute the value of a in equation (1) we get,
$(16 + 16r + 16r^2)/r = 56$
$16 + 16r + 16r^2 = 56r$
$16r^2 – 40r + 16 = 0$
$2r^2 – 5r + 2 = 0$
$2r^2 – 4r – r + 2 = 0$
$2r(r - 2) – 1(r - 2) = 0$
$(r - 2) (2r - 1) = 0$

r = 2 or 1/2
Substitute the value of r in equation (2) we get,
a = 16/r
= 16/2 or 16/(1/2)
= 8 or 32
∴ The three numbers are (a, ar, ar^2) is (8, 16, 32)

7. if a, b, c are in G.P., prove that:
(i) $a(b^2 + c^2) = c(a^2 + b^2)$
(ii) $a^2b^2c^2 [1/a^3 + 1/b^3 + 1/c^3] = a^3 + b^3 + c^3$
(iii) $(a+b+c)^2 / (a^2 + b^2 + c^2) = (a+b+c) / (a-b+c)$
(iv) $1/(a^2 – b^2) + 1/b^2 = 1/(b^2 – c^2)$
(v) $(a + 2b + 2c) (a – 2b + 2c) = a^2 + 4c^2$
Solution:
(i) $a(b^2 + c^2) = c(a^2 + b^2)$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $a(b^2 + c^2)$
Now, substituting $b^2 = ac$, we get
$a(ac + c^2)$
$a^2c + ac^2$
$c(a^2 + ac)$
Substitute $ac = b^2$ we get,
$c(a^2 + b^2)$ = RHS
∴ LHS = RHS
Hence proved.

(ii) $a^2b^2c^2 [1/a^3 + 1/b^3 + 1/c^3] = a^3 + b^3 + c^3$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $a^2b^2c^2 [1/a^3 + 1/b^3 + 1/c^3]$
$a^2b^2c^2/a^3 + a^2b^2c^2/b^3 + a^2b^2c^2/c^3$
$b^2c^2/a + a^2c^2/b + a^2b^2/c$
$(ac)c^2/a + (b^2)^2/b + a^2(ac)/c$ [by substituting the $b^2 = ac$]
$ac^3/a + b^4/b + a^3c/c$
$c^3 + b^3 + a^3$ = RHS
∴ LHS = RHS

Hence proved.

(iii) $(a+b+c)^2 / (a^2 + b^2 + c^2) = (a+b+c) / (a-b+c)$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $(a+b+c)^2 / (a^2 + b^2 + c^2)$
$(a+b+c)^2 / (a^2 + b^2 + c^2) = (a+b+c)^2 / (a^2 - b^2 + c^2 + 2b^2)$
$= (a+b+c)^2 / (a^2 - b^2 + c^2 + 2ac)$ [Since, $b^2 = ac$]
$= (a+b+c)^2 / (a+b+c)(a-b+c)$ [Since, $(a+b+c)(a-b+c) = a^2 - b^2 + c^2 + 2ac$]
$= (a+b+c) / (a-b+c)$
= RHS
∴ LHS = RHS
Hence proved.

(iv) $1/(a^2 - b^2) + 1/b^2 = 1/(b^2 - c^2)$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $1/(a^2 - b^2) + 1/b^2$
Let us take LCM
$1/(a^2 - b^2) + 1/b^2 = (b^2 + a^2 - b^2)/(a^2 - b^2)b^2$
$= a^2 / (a^2b^2 - b^4)$
$= a^2 / (a^2b^2 - (b^2)^2)$
$= a^2 / (a^2b^2 - (ac)^2)$ [Since, $b^2 = ac$]
$= a^2 / (a^2b^2 - a^2c^2)$
$= a^2 / a^2(b^2 - c^2)$
$= 1/ (b^2 - c^2)$
= RHS
∴ LHS = RHS
Hence proved.

(v) $(a + 2b + 2c) (a - 2b + 2c) = a^2 + 4c^2$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
Let us consider LHS: $(a + 2b + 2c) (a - 2b + 2c)$
Upon expansion we get,
$(a + 2b + 2c) (a - 2b + 2c) = a^2 - 2ab + 2ac + 2ab - 4b^2 + 4bc + 2ac - 4bc + 4c^2$

$= a^2 + 4ac – 4b^2 + 4c^2$
$= a^2 + 4ac – 4(ac) + 4c^2$ [Since, $b^2 = ac$]
$= a^2 + 4c^2$
= RHS

∴ LHS = RHS
Hence proved.

8. If a, b, c, d are in G.P., prove that:
(i) $(ab - cd) / (b^2 – c^2) = (a + c) / b$
(ii) $(a + b + c + d)^2 = (a + b)^2 + 2(b + c)^2 + (c + d)^2$
(iii) (b + c) (b + d) = (c + a) (c + d)
Solution:
(i) $(ab - cd) / (b^2 – c^2) = (a + c) / b$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
$bc = ad$
$c^2 = bd$
Let us consider LHS: $(ab - cd) / (b^2 – c^2)$
$(ab - cd) / (b^2 – c^2) = (ab - cd) / (ac - bd)$
$= (ab - cd)b / (ac - bd)b$
$= (ab^2 – bcd) / (ac - bd)b$
$= [a(ac) – c(c^2)] / (ac - bd)b$
$= (a^2c – c^3) / (ac - bd)b$
$= [c(a^2 – c^2)] / (ac - bd)b$
$= [(a+c) (ac – c^2)] / (ac - bd)b$
$= [(a+c) (ac - bd)] / (ac - bd)b$
$= (a+c) / b$
= RHS

∴ LHS = RHS
Hence proved.

(ii) $(a + b + c + d)^2 = (a + b)^2 + 2(b + c)^2 + (c + d)^2$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
$bc = ad$
$c^2 = bd$
Let us consider RHS: $(a + b)^2 + 2(b + c)^2 + (c + d)^2$

Let us expand

$(a + b)^2 + 2(b + c)^2 + (c + d)^2 = (a + b)^2 + 2\,(a+b)\,(c+d) + (c+d)^2$

$= a^2 + b^2 + 2ab + 2(c^2 + b^2 + 2cb) + c^2 + d^2 + 2cd$

$= a^2 + b^2 + c^2 + d^2 + 2ab + 2(c^2 + b^2 + 2cb) + 2cd$

$= a^2 + b^2 + c^2 + d^2 + 2(ab + bd + ac + cb + cd)$ [Since, $c^2 = bd$, $b^2 = ac$]

You can visualize the above expression by making separate terms for $(a + b + c)^2 + d^2 + 2d(a + b + c) = \{(a + b + c) + d\}^2$

∴ RHS = LHS

Hence proved.

(iii) $(b + c)\,(b + d) = (c + a)\,(c + d)$

Given that a, b, c are in GP.

By using the property of geometric mean,

$b^2 = ac$

$bc = ad$

$c^2 = bd$

Let us consider LHS: $(b + c)\,(b + d)$

Upon expansion we get,

$(b + c)\,(b + d) = b^2 + bd + cb + cd$

$= ac + c^2 + ad + cd$ [by using property of geometric mean]

$= c\,(a + c) + d\,(a + c)$

$= (a + c)\,(c + d)$

= RHS

∴ LHS = RHS

Hence proved.

9. If a, b, c are in G.P., prove that the following are also in G.P.:

(i) a^2, b^2, c^2

(ii) a^3, b^3, c^3

(iii) $a^2 + b^2, ab + bc, b^2 + c^2$

Solution:

(i) a^2, b^2, c^2

Given that a, b, c are in GP.

By using the property of geometric mean,

$b^2 = ac$

on squaring both the sides we get,

$(b^2)^2 = (ac)^2$

$(b^2)^2 = a^2c^2$

∴ a^2, b^2, c^2 are in G.P.

(ii) a^3, b^3, c^3
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
on squaring both the sides we get,
$(b^2)^3 = (ac)^3$
$(b^2)^3 = a^3c^3$
$(b^3)^2 = a^3c^3$
∴ a^3, b^3, c^3 are in G.P.

(iii) $a^2 + b^2, ab + bc, b^2 + c^2$
Given that a, b, c are in GP.
By using the property of geometric mean,
$b^2 = ac$
$a^2 + b^2, ab + bc, b^2 + c^2$ or $(ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$ [by using the property of GM]
Let us consider LHS: $(ab + bc)^2$
Upon expansion we get,
$(ab + bc)^2 = a^2b^2 + 2ab^2c + b^2c^2$
$= a^2b^2 + 2b^2(b^2) + b^2c^2$ [Since, $ac = b^2$]
$= a^2b^2 + 2b^4 + b^2c^2$
$= a^2b^2 + b^4 + a^2c^2 + b^2c^2$ {again using $b^2 = ac$ }
$= b^2(b^2 + a^2) + c^2(a^2 + b^2)$
$= (a^2 + b^2)(b^2 + c^2)$
= RHS
∴ LHS = RHS
Hence $a^2 + b^2, ab + bc, b^2 + c^2$ are in GP.