EXERCISE 18.3

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

(i) $f(x) = x^4 – 62x^2 + 120x + 9$

Solution:

Given $f(x) = x^4 – 62x^{2\,+}\ 120x + 9$

$\therefore f'(x) = 4x^3 – 124x + 120 = 4(x^3 – 31x + 30)$

$f''(x) = 12x^2 – 124 = 4(3x^2 – 31)$

For maxima and minima, $f'(x) = 0$

$4(x^3 – 31x + 30) = 0$

So roots will be $x = 5, 1, – 6$

Now, $f''(5) = 176 > 0$

$x = 5$ is point of local minima

$f''(1) = – 112 < 0$

$x = 1$ is point of local maxima

$f''(– 6) = 308 > 0$

$x = – 6$ is point of local minima

Local max value = $f(1) = 68$

Local min value = $f(5) = – 316$ and $f(– 6) = – 1647$

(ii) $f(x) = x^3 – 6x^2 + 9x + 15$

Solution:

Given $f(x) = x^3 – 6x^2 + 9x + 15$

Differentiating f with respect to x

$\therefore f'(x) = 3x^2 – 12x + 9 = 3(x^2 – 4x + 3)$

$f''(x) = 6x – 12 = 6(x – 2)$

For maxima and minima, $f'(x) = 0$

$3(x^2 – 4x + 3) = 0$

So roots will be $x = 3, 1$

Now, $f''(3) = 6 > 0$

$x = 3$ is point of local minima

$f''(1) = – 6 < 0$

$x = 1$ is point of local maxima

Local max value = f (1) = 19 and local min value = f (3) = 15

(iii) $f(x) = (x - 1)(x + 2)^2$

Solution:

Given $f(x) = (x - 1)(x + 2)^2$

$\therefore f'(x) = (x + 2)^2 + 2(x - 1)(x + 2)$

$= (x + 2)(x + 2 + 2x - 2)$

$= (x + 2)(3x)$

And $f''(x) = 3(x + 2) + 3x$

$= 6x + 6$

For maxima and minima, $f'(x) = 0$

$(x + 2)(3x) = 0$

So roots will be $x = 0, -2$

Now, $f''(0) = 6 > 0$

$x = 0$ is point of local minima

$f''(-2) = -6 < 0$

$x = -2$ is point of local maxima

Local maxima value = $f(-2) = 0$ and local minima value = $f(0) = -4$

(iv) $f(x) = 2/x - 2/x^2, x > 0$

Solution:

Given $f(x) = 2/x - 2/x^2, x > 0$

$$\therefore f'(x) = -\frac{2}{x^2} + \frac{4}{x^3}$$

And, $f''(x) = +\frac{4}{x^3} - \frac{12}{x^4}$

For maxima and minima, $f'(x) = 0$

$-2/x^2 + 4/x^3 = 0$

$-2(x - 2)/x^3 = 0$

$\Rightarrow (x - 2) = 0$

$\Rightarrow x = 2$

Now,

$f''(2) = 4/8 - 12/16 = ½ - ¾ = -¼ < 0$

x = 2 is point of local maxima

Local maxima value = f (2) = ½

(v) f (x) = x e^x

Solution:

Given f(x) = x e^x

f'(x) = e^x + x e^x = e^x(x + 1)

f''(x) = e^x(x + 1) + e^x

= e^x(x + 2)

For maxima and minima,

f'(x) = 0

e^x(x + 1) = 0

x = – 1

Now f'' (– 1) = e^{-1} = 1/e > 0

x = – 1 is point of local minima

Hence, local min = f (– 1) = – 1/e

2. Find the local extremum values of the following functions:

(i) f(x) = (x – 1) $(x – 2)^2$

Solution:

Given f(x) = (x – 1)$(x – 2)^2$

f'(x) = $(x – 2)^2$ + 2(x – 1)(x – 2)

= (x – 2) (x – 2 + 2x – 2)

= (x – 2) (3x – 4)

f''(x) = (3x – 4) + 3(x – 2)

For maxima and minima, f'(x) = 0

(x – 2)(3x – 4) = 0

x = 2, 4/3

Now f'' (2) > 0

x = 2 is point of local minima

f''(4/3) = – 2 < 0

x = 4/3 is point of local maxima

Hence local maxima value = f (4/3) = 4/27

Local minima value = f (2) = 0

(ii) $f(x) = x\sqrt{1-x}, x \le 1$

Solution:

Given $f(x) = x\sqrt{1-x}$

$\therefore f'(x) = \sqrt{1-x} + \frac{x}{2\sqrt{1-x}}(-1)$

$= \frac{2(1-x)-x}{2\sqrt{1-x}}$

$= \frac{2-3x}{2\sqrt{1-x}}$

f''(x) = $\frac{2\sqrt{(1-x)}(-3) + \frac{2-3x}{\sqrt{1-x}}}{4(1-x)}$

For maxima and minima, f'(x) = 0

$\frac{2-3x}{2\sqrt{1-x}} = 0$

x = 2/3

Now f'' (2/3) < 0

x = 2/3 is point of maxima

Hence local max value = f (2/3) = $\frac{2}{3\sqrt{3}}$

(iii) f (x) = – $(x-1)^3(x+1)^2$

Solution:

Given f (x) = – $(x-1)^3(x+1)^2$

f'(x) = – $3(x-1)^2(x+1)^2 - 2(x-1)^3(x+1)$

= – $(x-1)^2(x+1)$ (3x + 3 + 2x – 2)

= – $(x-1)^2(x+1)$ (5x + 1)

f''(x) = – 2(x – 1)(x + 1)(5x + 1) – $(x-1)^2(5x+1) - 5(x-1)^2(x+1)$

For maxima and minima, f'(x) = 0

$-(x-1)^2(x+1)(5x+1) = 0$
x = 1, – 1, – 1/5
Now f'' (1) = 0
x = 1 is inflection point
f''(– 1) = – 4× – 4 = 16 > 0
x = – 1 is point of minima
f'' (– 1/5) = – 5(36/25) × 4/5 = – 144/25 < 0
x = – 1/5 is point of maxima
Hence local max value = f (– 1/5) = 3456/3125
Local min value = f (– 1) = 0

3. The function y = a log x + bx^2 + x has extreme values at x = 1 and x = 2. Find a and b.

Solution:
Given y = a log x + bx^2 + x
On differentiating we get

$$\frac{dy}{dx} = \frac{a}{x} + 2bx + 1$$

And $\frac{d^2y}{dx^2} = -\frac{a}{x^2} + 2b$

For maxima and minima, $\frac{dy}{dx} = 0$

$$\frac{a}{x} + 2bx + 1 = 0$$

Given that extreme values exist at x = 1, 2
a + 2b = – 1 …… (1)

$$\frac{a}{2} + 4b = -1$$

a + 8b = – 2 …… (2)
Solving (1) and (2) we get
a = – 2/3 b = – 1/6

4. Show that log x/x has a maximum value at x = e.

Solution:

The given function is $f(x) = \frac{\log x}{x}$

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

Now f'(x) = 0

1 – Log x = 0

Log x = 1

Log x = log e

x = e

Now $f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$

$$= \frac{-x - 2x(1 - \log x)}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

Now $f''(e) = \frac{-3 + 2\log e}{e^3} = \frac{-3 + 2}{e^3} = -\frac{1}{e^3} < 0$ at x = e

Therefore, by second derivative test f attains the maximum value at x = e