

9. Continuity and Differentiability

Exercise 9A

1. Question

Show that $f(x) = x^2$ is continuous at $x=2$.

Answer

$$\begin{aligned}\text{Left Hand Limit: } \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} x^2 \\ &= 4\end{aligned}$$

$$\begin{aligned}\text{Right Hand Limit: } \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} x^2 \\ &= 4\end{aligned}$$

$$f(2) = 4$$

$$\text{Since, } \lim_{x \rightarrow 2} f(x) = f(2)$$

$\therefore f$ is continuous at $x=2$.

2. Question

Show that $f(x) = (x^2+3x+4)$ is continuous at $x=1$.

Answer

$$\begin{aligned}\text{Left Hand Limit: } \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} x^2 + 3x + 4 \\ &= 7\end{aligned}$$

$$\begin{aligned}\text{Right Hand Limit: } \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} x^2 + 3x + 4 \\ &= 7\end{aligned}$$

$$f(1) = 7$$

$$\text{Since, } \lim_{x \rightarrow 1} f(x) = f(1)$$

$\therefore f$ is continuous at $x=1$.

3. Question

Prove that

$$f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3}, & \text{when } x \neq 3 \\ 5, & \text{when } x = 3 \end{cases} \text{ is continuous at } x=3$$

Answer

$$\begin{aligned}\text{LHL: } \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} \frac{x^2 - x - 6}{x - 3} \\ &= \lim_{x \rightarrow 3^-} \frac{(x+2)(x-3)}{x-3} \text{ [By middle term splitting]} \\ &= \lim_{x \rightarrow 3^-} x + 2 \\ &= 5\end{aligned}$$

$$\text{RHL: } \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \frac{x^2 - x - 6}{x - 3}$$

$$= \lim_{x \rightarrow 3^-} \frac{(x+2)(x-3)}{x-3} \text{ [By middle term splitting]}$$

$$= \lim_{x \rightarrow 3^-} x + 2$$

$$= 5$$

$$f(3) = 5$$

$$\text{Since, } \lim_{x \rightarrow 3} f(x) = f(3)$$

\therefore f is continuous at $x=3$.

4. Question

Prove that

$$f(x) = \begin{cases} \frac{x^2 - 25}{x - 5}, & \text{when } x \neq 5 \\ 10, & \text{when } x = 5 \end{cases} \text{ is continuous at } x=5$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} \frac{x^2 - 25}{x - 5}$$

$$= \lim_{x \rightarrow 5^-} \frac{(x+5)(x-5)}{x-5} \text{ [By middle term splitting]}$$

$$= \lim_{x \rightarrow 5^-} x + 5$$

$$= 10$$

$$\text{RHL: } \lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} \frac{x^2 - 25}{x - 5}$$

$$= \lim_{x \rightarrow 5^+} \frac{(x+5)(x-5)}{x-5} \text{ [By middle term splitting]}$$

$$= \lim_{x \rightarrow 5^+} x + 5$$

$$= 10$$

$$f(5) = 10$$

$$\text{Since, } \lim_{x \rightarrow 5} f(x) = f(5)$$

\therefore f is continuous at $x=5$.

5. Question

Prove that

$$f(x) = \begin{cases} \frac{\sin 3x}{x}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases} \text{ is discontinuous at } x=0$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin 3x}{x}$$

$$= 3$$

$$[\lim_{x \rightarrow a} \frac{\sin nx}{x} = n]$$



$$\text{RHL: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\sin 3x}{x}$$

$$= 3$$

$$f(0) = 1$$

$$\text{Since, } \lim_{x \rightarrow 0} f(x) \neq f(0)$$

∴ f is discontinuous at x=0.

6. Question

Prove that

$$f(x) = \begin{cases} \frac{1 - \cos x}{x^2}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases} \text{ is discontinuous at } x=0$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1 - \cos x}{x^2}$$

$$= \lim_{x \rightarrow 0^-} \frac{2 \sin^2 \frac{x}{2}}{x^2}$$

$$= 2 \lim_{x \rightarrow 0^-} \frac{(\sin \frac{x}{2})^2}{x^2}$$

$$= 2 \times \frac{1}{4}$$

$$= \frac{1}{2}$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x^2}$$

$$= \lim_{x \rightarrow 0^+} \frac{2 \sin^2 \frac{x}{2}}{x^2}$$

$$= 2 \lim_{x \rightarrow 0^+} \frac{(\sin \frac{x}{2})^2}{x^2}$$

$$= 2 \times \frac{1}{4}$$

$$= \frac{1}{2}$$

$$f(0) = 1$$

$$\text{Since, } \lim_{x \rightarrow 0} f(x) \neq f(0)$$

∴ f is discontinuous at x=0.

7. Question

Prove that

$$f(x) = \begin{cases} 2 - x, & \text{when } x < 2; \\ 2 + x, & \text{when } x \geq 2 \end{cases} \text{ is discontinuous at } x=2$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 2 - x$$

$$= 4$$



$$\text{RHL: } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 2 - x$$

$$= 0$$

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

$\therefore f(x)$ is discontinuous at $x=2$

8. Question

Prove that

$$f(x) = \begin{cases} 3 - x, & \text{when } x \leq 0; \\ x^2, & \text{when } x > 0 \end{cases} \text{ is discontinuous at } x=0$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 3 - x$$

$$= 3$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2$$

$$= 0$$

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

$\therefore f(x)$ is discontinuous at $x=0$

9. Question

Prove that

$$f(x) = \begin{cases} 5x - 4, & \text{when } 0 < x \leq 1; \\ 4x^2 - 3x, & \text{when } 1 < x < 2 \end{cases} \text{ is continuous at } x=1$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 5x - 4$$

$$= 1$$

$$\text{RHL: } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 4x^2 - 3x$$

$$= 1$$

$f(x)=5x-4$ [this equation is taken as equality for $x=1$ lies there]

$$f(1)= 1$$

$$\text{Since, } \lim_{x \rightarrow 1} f(x) = f(1)$$

$\therefore f$ is continuous at $x=1$.

10. Question

Prove that

$$f(x) = \begin{cases} x - 1, & \text{when } 1 \leq x < 2; \\ 2x - 3, & \text{when } 2 \leq x \leq 3 \end{cases} \text{ is continuous at } x=2$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} x - 1$$

$$= 1$$

$$\text{RHL: } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 2x - 3$$

$$= 1$$

$f(x) = 2x - 3$ [this equation is taken as equality for $x = 1$ lies there]

$$f(2) = 1$$

$$\text{Since, } \lim_{x \rightarrow 2} f(x) = f(2)$$

$\therefore f$ is continuous at $x = 2$.

11. Question

Prove that

$$f(x) = \begin{cases} \cos x, & \text{when } x \geq 0; \\ -\cos x, & \text{when } x < 0 \end{cases} \text{ is discontinuous at } x = 0$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \cos x$$

$$= 1$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} -\cos x$$

$$= -1$$

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

$\therefore f(x)$ is discontinuous at $x = 0$



12. Question

Prove that

$$f(x) = \begin{cases} \frac{|x-a|}{x-a}, & \text{when } x \neq a; \\ 1, & \text{when } x = a \end{cases} \text{ is discontinuous at } x = a$$

Answer

$$\text{LHL: } \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} \frac{|x-a|}{x-a}$$

$$= \lim_{x \rightarrow a^-} \frac{-(x-a)}{x-a}$$

$$= -1$$

$$\text{RHL: } \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} \frac{|x-a|}{x-a}$$

$$= \lim_{x \rightarrow a^+} \frac{(x-a)}{x-a}$$

$$= 1$$

$$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$

$\therefore f(x)$ is discontinuous at $x = a$

13. Question

Prove that

$$f(x) = \begin{cases} \frac{1}{2}(x - |x|), & \text{when } x \neq 0; \\ 2, & \text{when } x = 0 \end{cases} \text{ is discontinuous at } x=0$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{2}(x - |x|)$$

$$= \lim_{x \rightarrow 0^-} \frac{1}{2}(x - (-x))$$

$$= \lim_{x \rightarrow 0^-} 2x$$

$$= 0$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{2}(x - |x|)$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{2}(x - (x))$$

$$= 0$$

$$f(0) = 2$$

$$\text{Since, } \lim_{x \rightarrow 0} f(x) \neq f(0)$$

∴ f is discontinuous at x=0.

14. Question

Prove that

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{when } x \neq 0; \\ 0, & \text{when } x = 0; \end{cases} \text{ is discontinuous at } x=0$$

Answer

$$\lim_{x \rightarrow 0} \sin \frac{1}{x} = 0$$

$\sin \frac{1}{x}$ is bounded function between -1 and +1.

$$\text{Also, } f(0) = 0$$

$$\text{Since, } \lim_{x \rightarrow 0} f(x) = f(0)$$

Hence, f is a continuous function.

15. Question

Prove that

$$f(x) = \begin{cases} 2x, & \text{when } x < 2; \\ 2, & \text{when } x = 2; \\ x^2, & \text{when } x > 2; \end{cases} \text{ is discontinuous at } x=2$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 2x$$

$$= 4$$

$$\text{RHL: } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} x^2$$

$$= 4$$

$$f(2) = 2$$

$$\text{Since, } \lim_{x \rightarrow 2} f(x) \neq f(2)$$

\therefore f is discontinuous at $x=2$.

16. Question

Prove that

$$f(x) = \begin{cases} -x, & \text{when } x < 0; \\ 1, & \text{when } x = 0; \\ x, & \text{when } x > 0; \end{cases} \text{ is discontinuous at } x=0$$

Answer

$$\text{LHL: } \lim_{x \rightarrow 0^-} f(x) = -x$$

$$= 0$$

$$\text{RHL: } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x$$

$$= 0$$

$$f(0) = 1$$

$$\text{Since, } \lim_{x \rightarrow 0} f(x) \neq f(0)$$

\therefore f is discontinuous at $x=0$.

17. Question

Find the value of k for which

$$f(x) = \begin{cases} \frac{\sin 2x}{5x}, & \text{when } x \neq 0; \\ \lambda, & \text{when } x = 0 \end{cases} \text{ is continuous at } x=0$$

Answer

Since, $f(x)$ is continuous at $x=0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = f(0)$$

$$\Rightarrow \frac{1}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \lambda$$

$$\Rightarrow \frac{1}{5} \times 2 = \lambda$$

$$\Rightarrow \lambda = \frac{2}{5}$$

18. Question

Find the value of λ for which

$$f(x) = \begin{cases} \frac{x^2 - 2x - 3}{x + 1}, & \text{when } x \neq -1; \\ \lambda, & \text{when } x = -1 \end{cases} \text{ is continuous at } x=-1$$

Answer

Since, $f(x)$ is continuous at $x=0$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{x + 1} = f(0)$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{(x-3)(x+1)}{x+1} = \lambda$$

$$\Rightarrow \lim_{x \rightarrow -1} x - 3 = \lambda$$

$$\Rightarrow \lambda = -4$$

19. Question

For what value of k is the following function continuous at $x=2$

$$f(x) = \begin{cases} 2x + 1, & \text{when } x < 2 \\ k, & \text{when } x = 2 \\ 3x - 1, & \text{when } x > 2 \end{cases}$$

Answer

Since, $f(x)$ is continuous at $x=2$

$$\Rightarrow \lim_{x \rightarrow 2^-} 2x + 1 = \lim_{x \rightarrow 2^+} 3x - 1 = f(2)$$

$$\Rightarrow \lim_{x \rightarrow 2^-} 2x + 1 = f(2)$$

$$\Rightarrow k = 5$$

20. Question

For what value of k is the following function

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & \text{when } x \neq 3; \\ k, & \text{when } x = 3 \end{cases} \text{ is continuous at } x=3$$

Ans. $k=6$

Answer

Since, $f(x)$ is continuous at $x=3$

$$\Rightarrow \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = f(3)$$

$$\Rightarrow \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} = f(3)$$

$$\Rightarrow \lim_{x \rightarrow 3} (x + 3) = f(3)$$

$$\Rightarrow k = 9$$

21. Question

For what value of k is the following function

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2}; \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \text{ is continuous at } x = \frac{\pi}{2}$$

Ans. $k=6$

Answer

f is continuous at $x = \frac{\pi}{2}$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = 3$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = 3 \text{ [Here } x = \frac{\pi}{2} - h \text{]}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{k \sin h}{\pi - \pi + 2h} = 3$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{k \sin h}{2h} = 3$$

$$\Rightarrow \frac{k}{2} \times 1 = 3$$

$$\Rightarrow k = 6$$

22. Question

Show that function:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases} \text{ is continuous at } x=0$$

Answer

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$$

As $\lim_{x \rightarrow 0} x^2 = 0$ and $\sin\left(\frac{1}{x}\right)$ is bounded function between -1 and $+1$.

$$\therefore \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

Also, $f(0) = 0$

Since, $\lim_{x \rightarrow 0} f(x) = f(0)$

Hence, f is a continuous function.

23. Question

$$\text{Show that: } f(x) = \begin{cases} x + 1, & \text{if } x \geq 1; \\ x^2 + 1, & \text{if } x < 1 \end{cases} \text{ is continuous at } x=1$$

Answer

$$\therefore \text{LHL: } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 + 1$$

$$= 2$$

$$\text{RHL: } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x + 1$$

$$= 2$$

$$f(1) = 2$$



Since, $\lim_{x \rightarrow 1} f(x) = f(1)$

$\therefore f$ is continuous at $x=1$.

24. Question

Show that: $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2; \\ x^2 + 1, & \text{if } x > 2 \end{cases}$ is continuous at $x=2$

Answer

$$\text{: LHL: } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} x^3 - 3$$
$$= 5$$

$$\text{RHL: } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x^2 + 1$$
$$= 5$$

$$f(2) = 5$$

Since, $\lim_{x \rightarrow 2} f(x) = f(2)$

$\therefore f$ is continuous at $x=2$.

25. Question

Find the values of a and b such that the following functions continuous. $\begin{cases} 5, & \text{when } x \leq 2 \\ ax + b, & \text{when } 2 < x < 10 \\ 21, & \text{when } x \geq 10 \end{cases}$

Answer

f is continuous at $x=2$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\lim_{x \rightarrow 2^-} (5) = \lim_{x \rightarrow 2^+} [ax + b] = 5$$

$$\Rightarrow 2a + b = 5 \dots\dots (1)$$

f is continuous at $x=10$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\lim_{x \rightarrow 2^-} (21) = \lim_{x \rightarrow 2^+} [ax + b] = 21$$

$$\Rightarrow 10a + b = 21 \dots\dots (2)$$

$$(1) - (2)$$

$$-8a = -16$$

$$a = 2$$

Putting a in 1

$$b = 1$$

26. Question

Find the values of a and b such that the following functions f, defined as $\begin{cases} a \sin \frac{\pi}{2}(x+1), & x \leq 0; \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$ is continuous

at $x=0$

Answer

: f is continuous at $x=0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$$

$$\lim_{x \rightarrow 0^-} (a \sin \frac{\pi}{2}(x+1)) = \lim_{x \rightarrow 0^+} \left[\frac{\tan x - \sin x}{x^3} \right]$$

$$(a \sin \frac{\pi}{2}(0+1)) = \lim_{x \rightarrow 0^+} \left[\frac{\sin x - \sin x}{\cos x x^3} \right]$$

$$a = \lim_{x \rightarrow 0^+} \left[\frac{\sin x (\frac{1}{\cos x} - 1)}{x^3} \right]$$

$$= \lim_{x \rightarrow 0^+} \left[\frac{\sin x (\frac{1}{\cos x} - 1)}{x^3} \right]$$

$$= \lim_{x \rightarrow 0^+} \left[\frac{\sin x (1 - \cos x)}{\cos x x^3} \right]$$

$$= \lim_{x \rightarrow 0^+} \left[\frac{\sin x \cdot 2 \sin^2 \frac{x}{2}}{\cos x x^3} \right]$$

$$= \lim_{x \rightarrow 0^+} \left[\frac{\sin x \cdot 2 \sin^2 \frac{x}{2}}{x \cdot x^2} \right] \times \frac{1}{\cos x}$$

$$= 1 \times 2 \times \frac{1}{4} \times 1$$

$$= \frac{1}{2}$$



27. Question

Prove that the function f given $f(x)=|x-3|$, $x \in \mathbb{R}$ is continuous but not differentiable at $x=3$

Answer

$$f(x)=|x-3|$$

Since every modulus function is continuous for all real x, f(x) is continuous at $x=3$.

$$f(x) = f(x) = \begin{cases} 3 - x, & x < 0 \\ x - 3, & x \geq 0 \end{cases}$$

To prove differentiable, we will use the following formula.

$$\lim_{x \rightarrow a^+} \frac{f(x)-f(a)}{x-a} = \lim_{x \rightarrow a^-} \frac{f(x)-f(a)}{x-a} = f'(a)$$

$$\text{L.H.L } \lim_{x \rightarrow a^+} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \rightarrow 3^+} \frac{x-3-0}{x-3}$$

$$= \lim_{x \rightarrow 3^+} \frac{x-3}{x-3}$$

$$= 1$$

$$\text{R.H.L: } \lim_{x \rightarrow a^-} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \rightarrow 3^+} \frac{3-x-0}{x-3}$$

$$= \lim_{x \rightarrow 3^+} \frac{3-x}{x-3}$$

$$= -1$$

Since, L.H.L \neq R.H.L, $f(x)$ is not differentiable at $x=5$.

Exercise 9B

1. Question

Show that function $f(x) = \begin{cases} (7x + 5), & \text{when } x \geq 0; \\ (5 - 3x), & \text{when } x < 0 \end{cases}$ is continuous function.

Answer

Given:

$$f(x) = \begin{cases} (7x + 5), & \text{when } x \geq 0; \\ (5 - 3x), & \text{when } x < 0 \end{cases}$$

Let's calculate the limit of $f(x)$ when x approaches 0 from the right

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (7x + 5) = 7(0) + 5$$

$$= 5$$

Therefore,

$$\lim_{x \rightarrow 0^+} f(x) = 5$$

Let's calculate the limit of $f(x)$ when x approaches 0 from the left

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (5 - 3x) = 5 - 3(0)$$

$$= 5$$

Therefore,

$$\lim_{x \rightarrow 0^-} f(x) = 5$$

Also, $f(0) = 5$

As we can see,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0) = 5$$

Thus, we can say that $f(x)$ is continuous function.

2. Question

Show that function $f(x) = \begin{cases} \sin x, & \text{if } x < 0; \\ x, & \text{if } x \geq 0 \end{cases}$ is continuous.

Answer

Given:

$$f(x) = \begin{cases} \sin x, & \text{if } x < 0; \\ x, & \text{if } x \geq 0 \end{cases}$$

Left hand limit at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (\sin x) = \sin(0) = 0$$

Therefore,

$$\lim_{x \rightarrow 0^-} f(x) = 0$$

Right hand limit at $x = 0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

Therefore,

$$\lim_{x \rightarrow 0^+} f(x) = 0$$

Also, $f(0) = 0$

As,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0) = 0$$

Thus, we can say that $f(x)$ is continuous function.

3. Question

Show that function $f(x) = \begin{cases} \frac{x^n - 1}{x - 1}, & \text{when } x \neq 1; \\ n, & \text{when } x = 1 \end{cases}$ is continuous.

Answer

Given:

$$f(x) = \begin{cases} \frac{x^n - 1}{x - 1}, & \text{when } x \neq 1; \\ n, & \text{when } x = 1 \end{cases}$$



Left hand limit and $x = 1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1 - h) = \lim_{h \rightarrow 0} \frac{(1-h)^n - 1}{(1-h) - 1}$$

$$\lim_{h \rightarrow 0} \frac{(1-h)^n - 1}{1 - h - 1} = \lim_{h \rightarrow 0} \frac{(1-h)^n - 1}{-h} = \lim_{h \rightarrow 0} -\frac{(1-h)^n - 1}{h}$$

$$= -\lim_{h \rightarrow 0} \frac{(1-h)^n - 1}{h} \quad (\text{Because } \lim_{x \rightarrow a} c \cdot f(x) = c \lim_{x \rightarrow a} f(x))$$

Applying L hospital's rule $\left(\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \right)$

$$= -\lim_{h \rightarrow 0} \frac{-n(1-h)^{n-1}}{1} = -[-n(1-0)^{n-1}] = n$$

Right hand limit and $x = 1$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1 + h) = \lim_{h \rightarrow 0} \frac{(1+h)^n - 1}{(1+h) - 1}$$

$$\lim_{h \rightarrow 0} \frac{(1+h)^n - 1}{1 + h - 1} = \lim_{h \rightarrow 0} \frac{(1+h)^n - 1}{h}$$

Applying L hospital's rule $\left(\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \right)$

$$= \lim_{h \rightarrow 0} \frac{n(1+h)^{n-1}}{1} = [n(1+0)^{n-1}] = n$$

Also, $f(x) = n$ at $x = 1$

As we can see that $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(x)$

Thus, $f(x)$ is continuous at $x = 1$

4. Question

Show that $\sec x$ is a continuous function.

Answer

Let $f(x) = \sec x$

Therefore, $f(x) = \frac{1}{\cos x}$

$f(x)$ is not defined when $\cos x = 0$

And $\cos x = 0$ when, $x = \frac{\pi}{2}$ and odd multiples of $\frac{\pi}{2}$ like $-\frac{\pi}{2}$

Let us consider the function

$f(a) = \cos a$ and let c be any real number. Then,

$$\lim_{a \rightarrow c^+} f(a) = \lim_{h \rightarrow 0} f(c + h)$$

$$\lim_{h \rightarrow 0} \cos(c + h) = \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h]$$

$$= \cos c \lim_{h \rightarrow 0} \cos h - \sin c \lim_{h \rightarrow 0} \sin h$$

$$= \cos c (1) - \sin c (0)$$

Therefore,

$$\lim_{a \rightarrow c^+} f(a) = \cos c$$

Similarly,

$$\lim_{a \rightarrow c^-} f(a) = f(c) = \cos c$$

Therefore,

$$\lim_{a \rightarrow c^-} f(a) = \lim_{a \rightarrow c^+} f(a) = f(c) = \cos c$$

So, $f(a)$ is continuous at $a = c$

Similarly, $\cos x$ is also continuous everywhere

Therefore, $\sec x$ is continuous on the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$

5. Question

Show that $\sec |x|$ is a continuous function

Answer

Let $f(x) = \sec |x|$ and a be any real number. Then,

Left hand limit at $x = a$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} \sec |x| = \lim_{h \rightarrow 0} \sec |a - h| = \sec |a|$$

Right hand limit at $x = a$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} \sec |x| = \lim_{h \rightarrow 0} \sec |a + h| = \sec |a|$$

Also, $f(a) = \sec |a|$



Therefore,

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

Thus, $f(x)$ is continuous at $x = a$.

6. Question

Show that function $f(x) = \begin{cases} (2-x), & \text{when } x \geq 1; \\ x, & \text{when } 0 \leq x \leq 1. \end{cases}$ is continuous.

Answer

We know that $\sin x$ is continuous everywhere

Consider the point $x = 0$

Left hand limit:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(\frac{\sin x}{x} \right) = \lim_{h \rightarrow 0} \left(\frac{\sin(0-h)}{0-h} \right) = \lim_{h \rightarrow 0} \left(\frac{-\sin h}{-h} \right) = 1$$

Right hand limit:

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} \right) = \lim_{h \rightarrow 0} \left(\frac{\sin(0+h)}{0+h} \right) = \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) = 1$$

Also we have,

$$f(0) = 2$$

As,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) \neq f(0)$$

Therefore, $f(x)$ is discontinuous at $x = 0$.

7. Question

Discuss the continuity of $f(x) = [x]$.

Answer

Let n be any integer

$[x]$ = Greatest integer less than or equal to x .

Some values of $[x]$ for specific values of x

$$[3] = 3$$

$$[4.4] = 4$$

$$[-1.6] = -2$$

Therefore,

Left hand limit at $x = n$

$$\lim_{x \rightarrow n^-} f(x) = \lim_{x \rightarrow n^-} [x] = n - 1$$

Right hand limit at $x = n$

$$\lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} [x] = n$$

Also, $f(n) = [n] = n$

$$\text{As } \lim_{x \rightarrow n^-} f(x) \neq \lim_{x \rightarrow n^+} f(x)$$



Therefore, $f(x) = [x]$ is discontinuous at $x = n$.

8. Question

Show that $f(x) = \begin{cases} (2x - 1), & \text{if } x < 2; \\ \frac{3x}{2}, & \text{if } x \geq 2 \end{cases}$ is continuous.

Answer

Given function $f(x) = \begin{cases} (2x - 1), & \text{if } x < 2; \\ \frac{3x}{2}, & \text{if } x \geq 2 \end{cases}$

Left hand limit at $x = 2$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2} (2x - 1) = 2(2) - 1 = 3$$

Right hand limit at $x = 2$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2} \frac{3x}{2} = \frac{3(2)}{2} = 3$$

Also,

$$f(2) = \frac{3(2)}{2} = 3$$

As

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2) = 3$$

Therefore,

The function $f(x)$ is continuous at $x = 2$.

9. Question

Show that $f(x) = \begin{cases} x, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at each point except 0.

Answer

Given function is $f(x) = \begin{cases} x, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0 \end{cases}$

Left hand limit at $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} f(-h) = 0$$

Right hand limit at $x = 0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} f(h) = 0$$

Also,

$$f(0) = 1$$

As,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) \neq f(0)$$

$f(x) = x$ for other values of x except 0 $f(x) = 1, 2, 3, 4, \dots$

Therefore,

$f(x)$ is not continuous everywhere except at $x = 0$

10. Question

Locate the point of discontinuity of the function

$$f(x) = \begin{cases} (x^3 - x^2 + 2x - 2), & \text{if } x \neq 1; \\ 4, & \text{if } x = 1 \end{cases}$$

Answer

$$\text{Given function } f(x) = \begin{cases} (x^3 - x^2 + 2x - 2), & \text{if } x \neq 1; \\ 4, & \text{if } x = 1 \end{cases}$$

$$\text{Left hand limit at } x = 1: \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^3 - x^2 + 2x - 2)$$

$$= \lim_{h \rightarrow 0} \{(1 - h)^3 - (1 - h)^2 + 2(1 - h) - 2\}$$

$$= \lim_{h \rightarrow 0} (1 - h)^3 - \lim_{h \rightarrow 0} (1 - h)^2 + 2 \lim_{h \rightarrow 0} (1 - h) - 2$$

$$= 1 - 1 + 2 - 2$$

$$= 0$$

$$\text{Right hand limit at } x = 1: \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^3 - x^2 + 2x - 2)$$

$$= \lim_{h \rightarrow 0} \{(1 + h)^3 - (1 + h)^2 + 2(1 + h) - 2\}$$

$$= \lim_{h \rightarrow 0} (1 + h)^3 - \lim_{h \rightarrow 0} (1 + h)^2 + 2 \lim_{h \rightarrow 0} (1 + h) - 2$$

$$= 1 - 1 + 2 - 2$$

$$= 0$$

$$\text{Also, } f(1) = 4$$

As we can see that,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) \neq f(1)$$

Therefore,

$f(x)$ is not continuous at $x = 1$

11. Question

Discuss the continuity of the function $f(x) = |x| + |x - 1|$ in the interval of $[-1, 2]$

Answer

$$\text{Given function } f(x) = |x| + |x - 1|$$

A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if and only if,

(i) f is continuous on the open interval (a, b)

$$\text{(ii) } \lim_{x \rightarrow a^+} f(x) = f(a)$$

$$\text{(iii) } \lim_{x \rightarrow b^-} f(x) = f(b)$$

Let's check continuity on the open interval $(-1, 2)$

As $-1 < x < 2$

Left hand limit:

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{h \rightarrow 0} \{|-1 - h| + |(-1 - h) - 1|\}$$



$$= |-1-0| + |(-1-0) - 1|$$

$$= 1 + 2$$

$$= 3$$

Right hand limit:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} \{|2 + h| + |(2 + h) - 1|\}$$

$$= |2| + |2 - 1|$$

$$= 2 + 1$$

$$= 3$$

Left hand limit = Right hand limit

Here $a = -1$ and $b = 2$

Therefore,

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{h \rightarrow 0} \{|-1 + h| + |(-1 + h) - 1|\}$$

$$= |-1 + 0| + |(-1 + 0) - 1|$$

$$= |-1| + |-1 - 1|$$

$$= 1 + 2 = 3$$

$$\text{Also } f(-1) = |-1| + |-1 - 1| = 1 + 2 = 3$$

Now,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} \{|2 - h| + |(2 - h) - 1|\}$$

$$= |2 - 0| + |(2 - 0) - 1|$$

$$= |2| + |2 - 1|$$

$$= 2 + 1 = 3$$

$$\text{Also } f(2) = |2| + |2 - 1| = 2 + 1 = 3$$

Therefore,

$f(x)$ is continuous on the closed interval $[-1, 2]$.

Exercise 9C

1. Question

Show that $f(x) = x^3$ is continuous as well as differentiable at $x=3$.

Answer

Given:

$$f(x) = x^3$$

If a function is differentiable at a point, it is necessarily continuous at that point.

Left hand derivative (LHD) at $x = 3$

$$\lim_{x \rightarrow 3^-} \frac{f(x) - f(3)}{x - 3} = \lim_{h \rightarrow 0} \frac{f(3-h) - f(3)}{(3-h) - 3}$$

$$= \lim_{h \rightarrow 0} \frac{(3-h)^3 - 3^3}{(3-h) - 3} = \lim_{h \rightarrow 0} \frac{(3-h)^3 - 27}{-h} = \lim_{h \rightarrow 0} - \frac{h\{(3-h)^2 + 3(3-h) + 9\}}{h}$$

$$= \lim_{h \rightarrow 0} - \{(3-h)^2 + 3(3-h) + 9\} = \lim_{h \rightarrow 0} - [-\{(3-h)^2 - 3(3-h) - 9\}]$$



$$= \lim_{h \rightarrow 0} -\{-h^2 + 9h - 27\} = \lim_{h \rightarrow 0} h^2 - 9h + 27 = 0^2 - 9(0) + 27 = 27$$

Right hand derivative (RHD) at $x = 3$

$$\begin{aligned} \lim_{x \rightarrow 3^+} \frac{f(x) - f(3)}{x - 3} &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{(3+h) - 3} \\ &= \lim_{h \rightarrow 0} \frac{(3+h)^3 - 3^3}{(3+h) - 3} = \lim_{h \rightarrow 0} \frac{(3+h)^3 - 27}{h} = \lim_{h \rightarrow 0} \frac{h\{(3+h)^2 + 3(3+h) + 9\}}{h} \\ &= \lim_{h \rightarrow 0} \{(3+h)^2 + 3(3+h) + 9\} = \lim_{h \rightarrow 0} (3+h)^2 + 3(3+h) + 9 \\ &= \lim_{h \rightarrow 0} \{h^2 + 9h + 27\} = 0^2 + 9(0) + 27 = 27 \end{aligned}$$

LHD = RHD

Therefore, $f(x)$ is differentiable at $x = 3$.

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} x^3 = 3^3 = 27$$

Also, $f(3) = 27$

Therefore, $f(x)$ is also continuous at $x = 3$.

2. Question

Show that $f(x) = (x-1)^{1/3}$ is not differentiable at $x=1$.

Answer

Given function $f(x) = (x-1)^{1/3}$

LHD at $x = 1$

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{(1-h) - 1} = \lim_{h \rightarrow 0} \frac{\{(1-h) - 1\}^{\frac{1}{3}} - \{(1-1) - 1\}^{\frac{1}{3}}}{(1-h) - 1} \\ &= \lim_{h \rightarrow 0} \frac{(-h)^{\frac{1}{3}} - 0}{-h} = \frac{0}{0} = \text{Not defined} \end{aligned}$$

RHD at $x = 1$

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{(1+h) - 1} = \lim_{h \rightarrow 0} \frac{\{(1+h) - 1\}^{\frac{1}{3}} - \{(1-1) - 1\}^{\frac{1}{3}}}{(1+h) - 1} \\ &= \lim_{h \rightarrow 0} \frac{(h)^{\frac{1}{3}} - 0}{h} = \frac{0}{0} = \text{Not defined} \end{aligned}$$

Since, LHD and RHD doesn't exist

Therefore, $f(x)$ is not differentiable at $x = 1$.

3. Question

Show that constant function is always differentiable

Answer

Let a be any constant number.

Then, $f(x) = a$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

We know that coefficient of a linear function is

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

Since our function is constant, $y_1 = y_2$

Therefore, $a = 0$

Now,

$$f'(x) = \lim_{h \rightarrow 0} \frac{a-a}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} 0 = 0$$

Thus, the derivative of a constant function is always 0.

4. Question

Show that $f(x) = |x-5|$ is continuous but not differentiable at $x=5$

Answer

Left hand limit at $x = 5$

$$\lim_{x \rightarrow 5^-} |x - 5| = \lim_{x \rightarrow 5} (5 - x) = 0$$

Right hand limit at $x = 5$

$$\lim_{x \rightarrow 5^+} |x - 5| = \lim_{x \rightarrow 5} (x - 5) = 0$$

$$\text{Also } f(5) = |5 - 5| = 0$$

As,

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x) = f(5)$$

Therefore, $f(x)$ is continuous at $x = 5$

Now, let's see the differentiability of $f(x)$

LHD at $x = 5$

$$\lim_{x \rightarrow 5^-} \frac{f(x)-f(5)}{x-5} = \lim_{h \rightarrow 0} \frac{f(5-h)-f(5)}{5-h-5} = \lim_{h \rightarrow 0} \frac{|5-(5-h)|-|5-5|}{-h} = \lim_{h \rightarrow 0} -\frac{h}{h} = -1$$

RHD at $x = 5$

$$\lim_{x \rightarrow 5^+} \frac{f(x)-f(5)}{x-5} = \lim_{h \rightarrow 0} \frac{f(5+h)-f(5)}{5+h-5} = \lim_{h \rightarrow 0} \frac{|(5+h)-5|-|5-5|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

Since, $LHD \neq RHD$

Therefore,

$f(x)$ is not differentiable at $x = 5$

5. Question

$$\text{Let } f(x) = \begin{cases} (2-x), & \text{when } x \geq 1; \\ x, & \text{when } 0 \leq x < 1. \end{cases}$$

Show that $f(x)$ is continuous but not differentiable at $x=1$

Answer

Left hand limit at $x = 1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} x = 1$$

$f(x) = x$ is polynomial function and a polynomial function is continuous everywhere

Right hand limit at $x = 1$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} (2-x) = (2-1) = 1$$

$f(x) = 2 - x$ is polynomial function and a polynomial function is continuous everywhere

Also, $f(1) = 1$

As we can see that,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

Therefore,

$f(x)$ is continuous at $x = 1$

Now,

LHD at $x = 1$

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x - 1}{x - 1} = \lim_{x \rightarrow 1} 1 = 1$$

RHD at $x = 1$

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{2 - x - (2 - 1)}{x - 1} = \lim_{x \rightarrow 1} \frac{2 - x - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{-(x - 1)}{x - 1}$$

$$\lim_{x \rightarrow 1} -\frac{1}{1} = \lim_{x \rightarrow 1} -1 = -1$$

As, LHD \neq RHD

Therefore,

$f(x)$ is not differentiable at $x = 1$

6. Question

Show that $f(x) = [x]$ is neither continuous nor derivable at $x = 2$.

Answer

Left hand limit at $x = 2$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2 - h) = \lim_{h \rightarrow 0} [2 - h] = \lim_{h \rightarrow 0} 1 = 1$$

Right hand limit at $x = 2$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2 + h) = \lim_{h \rightarrow 0} [2 + h] = \lim_{h \rightarrow 0} 2 = 2$$

As left hand limit \neq right hand limit

Therefore, $f(x)$ is not continuous at $x = 2$

Lets see the differentiability of $f(x)$:

LHD at $x = 2$

$$\begin{aligned} \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} &= \lim_{h \rightarrow 0} \frac{f(x - h) - f(2)}{(x - h) - 2} = \lim_{h \rightarrow 0} \frac{f(2 - h) - f(2)}{(2 - h) - 2} \\ &= \lim_{h \rightarrow 0} -\frac{1 - 2}{h} \end{aligned}$$

$$\lim_{h \rightarrow 0} -\frac{(-1)}{h} = \infty$$

RHD at $x = 2$

$$\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(2)}{(x + h) - 2} = \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{(2 + h) - 2} = \lim_{h \rightarrow 0} \frac{2 - 2}{h}$$

$$\lim_{h \rightarrow 0} \frac{0}{h} = 0$$

As, LHD \neq RHD

Therefore,

$f(x)$ is not derivable at $x = 2$

7. Question

Show that function

$$f(x) = \begin{cases} (1-x), & \text{when } x < 1; \\ (x^2 - 1), & \text{when } x \geq 1. \end{cases} \text{ is continuous but not differentiable at } x=1$$

Answer

$$\text{Given function } f(x) = \begin{cases} (1-x), & \text{when } x < 1; \\ (x^2 - 1), & \text{when } x \geq 1. \end{cases}$$

Left hand limit at $x = 1$:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (1-x) = 1 - 1 = 0$$

Right hand limit at $x = 1$:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2 - 1) = 1^2 - 1 = 0$$

$$\text{Also, } f(1) = 1^2 - 1 = 0$$

As,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

Therefore,

$f(x)$ is continuous at $x = 1$

Now, let's see the differentiability of $f(x)$:

LHD at $x = 2$:

$$\begin{aligned} \lim_{x \rightarrow 2^-} \frac{f(x)-f(2)}{x-2} &= \lim_{x \rightarrow 2^-} \frac{(1-x)-(1-2)}{x-2} = \lim_{x \rightarrow 2^-} \frac{1-x-1+2}{x-2} = \lim_{x \rightarrow 2^-} \frac{-(x-2)}{x-2} \\ &= \lim_{x \rightarrow 2^-} -1 = -1 \end{aligned}$$

RHD at $x = 2$:

$$\begin{aligned} \lim_{x \rightarrow 2^+} \frac{f(x)-f(2)}{x-2} &= \lim_{x \rightarrow 2^+} \frac{(x^2-1)-(2^2-1)}{x-2} = \lim_{x \rightarrow 2^+} \frac{x^2-1-3}{x-2} = \lim_{x \rightarrow 2^+} \frac{x^2-4}{x-2} \\ &= \lim_{x \rightarrow 2^+} \frac{x^2-2^2}{x-2} = \lim_{x \rightarrow 2^+} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2^+} (x+2) = 2 + 2 = 4 \end{aligned}$$

As, LHD \neq RHD

Therefore,

$f(x)$ is not differentiable at $x = 2$

8. Question

$$\text{Let } f(x) = \begin{cases} (2+x), & \text{if } x \geq 0; \\ (2-x), & \text{if } x < 0. \end{cases} \text{ Show that } f(x) \text{ is not derivable at } x=0.$$

Answer

$$\text{Given function } f(x) = \begin{cases} (2+x), & \text{if } x \geq 0; \\ (2-x), & \text{if } x < 0. \end{cases}$$

LHD at $x = 0$:

$$\begin{aligned}\lim_{x \rightarrow 0^-} \frac{f(x)-f(0)}{x-0} &= \lim_{x \rightarrow 0} \frac{(2-x)-(2)}{x-0} = \lim_{x \rightarrow 0} \frac{-x}{x} \\ &= \lim_{x \rightarrow 0} -1 = -1\end{aligned}$$

RHD at $x = 0$:

$$\lim_{x \rightarrow 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \rightarrow 0} \frac{(2+x)-(2)}{x-0} = \lim_{x \rightarrow 0} \frac{x}{x} = \lim_{x \rightarrow 0} 1 = 1$$

As, LHD \neq RHD

Therefore,

$f(x)$ is not differentiable at $x = 0$

9. Question

If $f(x) = |x|$ show that $f'(2)=1$

Answer

Given function is $f(x) = |x|$

LHD at $x = 2$:

$$\begin{aligned}\lim_{x \rightarrow 2^-} \frac{f(x)-f(2)}{x-2} &= \lim_{h \rightarrow 0} \frac{f(2-h)-f(2)}{2-h-2} = \lim_{h \rightarrow 0} \frac{|2-h|-|2|}{-h} = \lim_{h \rightarrow 0} \frac{-h}{-h} \\ &= \lim_{h \rightarrow 0} 1 = 1\end{aligned}$$

RHD at $x = 2$:

$$\begin{aligned}\lim_{x \rightarrow 2^+} \frac{f(x)-f(2)}{x-2} &= \lim_{h \rightarrow 0} \frac{f(2+h)-f(2)}{2+h-2} = \lim_{h \rightarrow 0} \frac{|2+h|-|2|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} \\ &= \lim_{h \rightarrow 0} 1 = 1\end{aligned}$$

As, LHD = RHD

Therefore, $f(x) = |x|$ is differentiable at $x = 2$

$$\text{Now } f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h)-f(2)}{h} = \lim_{h \rightarrow 0} \frac{|2+h|-|2|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1$$

Therefore,

$$f'(2) = 1$$

10. Question

Find the values of a and b so that the function

$$f(x) = \begin{cases} (x^2 + 3x + a), & \text{when } x \leq 1; \\ (bx + 2), & \text{when } x > 1 \end{cases} \text{ is differentiable at each } x \in \mathbb{R}$$

Answer

It is given that $f(x)$ is differentiable at each $x \in \mathbb{R}$

For $x \leq 1$,

$$f(x) = x^2 + 3x + a \text{ i.e. a polynomial}$$

for $x > 1$,

$$f(x) = bx + 2, \text{ which is also a polynomial}$$

Since, a polynomial function is everywhere differentiable. Therefore, $f(x)$ is differentiable for all $x > 1$ and for all $x < 1$.

$f(x)$ is continuous at $x = 1$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

$$\lim_{x \rightarrow 1} (x^2 + 3x + a) = \lim_{x \rightarrow 1} (bx + 2) = 1 + 3 + a$$

$$1^2 + 3(1) + a = b(1) + 2 = 4 + a$$

$$4 + a = b + 2$$

$$a - b + 2 = 0 \dots(1)$$

As function is differentiable, therefore, LHD = RHD

LHD at $x = 1$:

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^-} \frac{x^2 + 3x + a - (4 + a)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{x^2 + 3x - 4}{x - 1} = \lim_{x \rightarrow 1^-} \frac{(x + 4)(x - 1)}{x - 1} \\ &= \lim_{x \rightarrow 1^-} (x + 4) = 1 + 4 = 5 \end{aligned}$$

RHD at $x = 1$:

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{(bx + 2) - (4 + a)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{bx - 2 - a}{x - 1} = \lim_{x \rightarrow 1^+} \frac{bx - b}{x - 1} = \lim_{x \rightarrow 1^+} \frac{b(x - 1)}{x - 1} \\ &= \lim_{x \rightarrow 1^+} b = b \end{aligned}$$

As, LHD = RHD

Therefore,

$$5 = b$$

Putting b in (1), we get,

$$a - b + 2 = 0$$

$$a - 5 + 2 = 0$$

$$a = 3$$

Hence,

$$a = 3 \text{ and } b = 5$$

