EXERCISE 23.13

1. Find the angles between each of the following pairs of straight lines:
(i) 3x + y + 12 = 0 and x + 2y – 1 = 0
(ii) 3x – y + 5 = 0 and x – 3y + 1 = 0
Solution:
(i) 3x + y + 12 = 0 and x + 2y – 1 = 0
Given:
The equations of the lines are
3x + y + 12 = 0 … (1)
x + 2y − 1 = 0 … (2)
Let m_1 and m_2 be the slopes of these lines.
$m_1 = -3$, $m_2 = -1/2$
Let θ be the angle between the lines.
Then, by using the formula
$\tan\theta = [(m_1 – m_2) / (1 + m_1m_2)]$
$= [(-3 + 1/2) / (1 + 3/2)]$
$= 1$
So,
$\theta = \pi/4$ or 45^o
∴ The acute angle between the lines is 45°

(ii) 3x – y + 5 = 0 and x – 3y + 1 = 0
Given:
The equations of the lines are
3x − y + 5 = 0 … (1)
x − 3y + 1 = 0 … (2)
Let m_1 and m_2 be the slopes of these lines.
$m_1 = 3$, $m_2 = 1/3$
Let θ be the angle between the lines.
Then, by using the formula
$\tan\theta = [(m_1 – m_2) / (1 + m_1m_2)]$
$= [(3 - 1/3) / (1 + 3(1/3))]$
$= [((9 - 1)/3)/(1 + 1)]$
$= 8/6$
$= 4/3$
So,
$\theta = \tan^{-1} (4/3)$
∴ The acute angle between the lines is $\tan^{-1} (4/3)$.

2. Find the acute angle between the lines 2x – y + 3 = 0 and x + y + 2 = 0.
Solution:
Given:
The equations of the lines are
$2x - y + 3 = 0$ … (1)
$x + y + 2 = 0$ … (2)
Let m_1 and m_2 be the slopes of these lines.
$m_1 = 2$, $m_2 = -1$
Let θ be the angle between the lines.
Then, by using the formula
$\tan \theta = [(m_1 - m_2) / (1 + m_1 m_2)]$
$= [(2 - (-1) / (1 + (2)(-1))]$
$= [3/(1 - 2)]$
$= 3$
So,
$\theta = \tan^{-1} (3)$
∴ The acute angle between the lines is $\tan^{-1} (3)$.

3. Prove that the points (2, -1), (0, 2), (2, 3) and (4, 0) are the coordinates of the vertices of a parallelogram and find the angle between its diagonals.
Solution:
To prove:
The points (2, -1), (0, 2), (2, 3) and (4, 0) are the coordinates of the vertices of a parallelogram
Let us assume the points, A (2, − 1), B (0, 2), C (2, 3) and D (4, 0) be the vertices.
Now, let us find the slopes

Slope of AB = [(2+1) / (0-2)]
= -3/2

Slope of BC = [(3-2) / (2-0)]
= ½

Slope of CD = [(0-3) / (4-2)]
= -3/2
Slope of DA = [(-1-0) / (2-4)]
= ½
Thus, AB is parallel to CD and BC is parallel to DA.
Hence proved, the given points are the vertices of a parallelogram.

Now, let us find the angle between the diagonals AC and BD.
Let m_1 and m_2 be the slopes of AC and BD, respectively.
$m_1 = [(3+1) / (2-2)]$
$= \infty$

$m_2 = [(0-2) / (4-0)]$
$= -1/2$

Thus, the diagonal AC is parallel to the y-axis.
$\angle ODB = \tan^{-1} (1/2)$

In triangle MND,
$\angle DMN = \pi/2 - \tan^{-1} (1/2)$
∴ The angle between the diagonals is $\pi/2 - \tan^{-1} (1/2)$.

4. Find the angle between the line joining the points (2, 0), (0, 3) and the line x + y = 1.
Solution:
Given:
Points (2, 0), (0, 3) and the line $x + y = 1$.
Let us assume A (2, 0), B (0, 3) be the given points.
Now, let us find the slopes
Slope of AB = m_1
$= [(3-0) / (0-2)]$
$= -3/2$

Slope of the line $x + y = 1$ is -1
∴ $m_2 = -1$

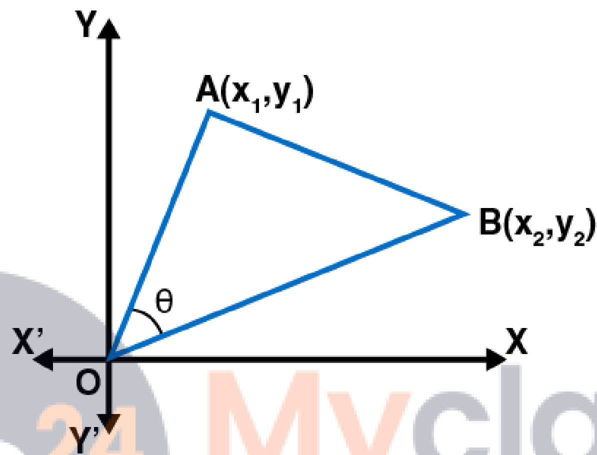
Let θ be the angle between the line joining the points (2, 0), (0, 3) and the line x + y =
$\tan \theta = |[(m_1 – m_2) / (1 + m_1m_2)]|$
$= [(-3/2 + 1) / (1 + 3/2)]$
$= 1/5$
$\theta = \tan^{-1} (1/5)$
∴ The acute angle between the line joining the points (2, 0), (0, 3) and the line $x + y = 1$ is $\tan^{-1} (1/5)$.

5. If θ is the angle which the straight line joining the points (x_1, y_1) and (x_2, y_2) subtends at the origin, prove that $\tan\theta = \frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2}$ **and** $\cos\theta = \frac{x_1y_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}$

Solution:

We need to prove:

$$\tan\theta = \frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2} \text{ and } \cos\theta = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.$$



Let us assume A (x_1, y_1) and B (x_2, y_2) be the given points and O be the origin.

Slope of OA = $m_1 = y_{1x1}$

Slope of OB = $m_2 = y_{2x2}$

It is given that θ is the angle between lines OA and OB.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Now, substitute the values, we get

$$= \frac{\frac{y_1}{x_1} - \frac{y_2}{x_2}}{1 + \frac{y_1}{x_1} \times \frac{y_2}{x_2}}$$

$$\tan\theta = \frac{x_2 y_1 - x_1 y_2}{x_1 x_2 + y_1 y_2}$$

Now,

As we know that $\cos\theta = \sqrt{\frac{1}{1 + \tan^2\theta}}$

Now, substitute the values, we get

$$\cos\theta = \frac{x_1 x_2 + y_1 y_2}{\sqrt{(x_2 y_1 - x_1 y_2)^2 + (x_1 x_2 + y_1 y_2)^2}}$$

$$\cos\theta = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 y_1^2 + x_1^2 y_2^2 + x_1^2 x_2^2 + y_1^2 y_2^2}}$$

$$\cos\theta = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.$$

Hence proved.