

EXERCISE 8.2

1. Express each of the following as the product of sines and cosines:

(i) $\sin 12x + \sin 4x$

(ii) $\sin 5x - \sin x$

(iii) $\cos 12x + \cos 8x$

(iv) $\cos 12x - \cos 4x$

(v) $\sin 2x + \cos 4x$

Solution:

(i) $\sin 12x + \sin 4x$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned}\sin 12x + \sin 4x &= 2 \sin \frac{(12x + 4x)}{2} \cos \frac{(12x - 4x)}{2} \\ &= 2 \sin 8x \cos 4x\end{aligned}$$

(ii) $\sin 5x - \sin x$

By using the formula,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\begin{aligned}\sin 5x - \sin x &= 2 \cos \frac{(5x + x)}{2} \sin \frac{(5x - x)}{2} \\ &= 2 \cos 3x \sin 2x\end{aligned}$$

(iii) $\cos 12x + \cos 8x$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned}\cos 12x + \cos 8x &= 2 \cos \frac{(12x + 8x)}{2} \cos \frac{(12x - 8x)}{2} \\ &= 2 \cos 10x \cos 2x\end{aligned}$$

(iv) $\cos 12x - \cos 4x$

By using the formula,

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\begin{aligned}\cos 12x - \cos 4x &= -2 \sin \frac{(12x + 4x)}{2} \sin \frac{(12x - 4x)}{2} \\ &= -2 \sin 8x \sin 4x\end{aligned}$$

(v) $\sin 2x + \cos 4x$

$$\sin 2x + \cos 4x = \sin 2x + \sin (90^\circ - 4x)$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned} \sin 2x + \sin (90^\circ - 4x) &= 2 \sin \frac{(2x + 90^\circ - 4x)}{2} \cos \frac{(2x - 90^\circ + 4x)}{2} \\ &= 2 \sin \frac{(90^\circ - 2x)}{2} \cos \frac{(6x - 90^\circ)}{2} \\ &= 2 \sin (45^\circ - x) \cos (3x - 45^\circ) \\ &= 2 \sin (45^\circ - x) \cos \{-(45^\circ - 3x)\} \text{ (since, } \{\cos (-x) = \cos x\}) \\ &= 2 \sin (45^\circ - x) \cos (45^\circ - 3x) \\ &= 2 \sin (\pi/4 - x) \cos (\pi/4 - 3x) \end{aligned}$$

2. Prove that :

(i) $\sin 38^\circ + \sin 22^\circ = \sin 82^\circ$

(ii) $\cos 100^\circ + \cos 20^\circ = \cos 40^\circ$

(iii) $\sin 50^\circ + \sin 10^\circ = \cos 20^\circ$

(iv) $\sin 23^\circ + \sin 37^\circ = \cos 7^\circ$

(v) $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$

(vi) $\sin 40^\circ + \sin 20^\circ = \cos 10^\circ$

Solution:

(i) $\sin 38^\circ + \sin 22^\circ = \sin 82^\circ$

Let us consider LHS:

$$\sin 38^\circ + \sin 22^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned} \sin 38^\circ + \sin 22^\circ &= 2 \sin \frac{(38^\circ + 22^\circ)}{2} \cos \frac{(38^\circ - 22^\circ)}{2} \\ &= 2 \sin 60^\circ/2 \cos 16^\circ/2 \\ &= 2 \sin 30^\circ \cos 8^\circ \\ &= 2 \times 1/2 \times \cos 8^\circ \\ &= \cos 8^\circ \\ &= \cos (90^\circ - 82^\circ) \\ &= \sin 82^\circ \text{ (since, } \{\cos (90^\circ - A) = \sin A\}) \\ &= \text{RHS} \end{aligned}$$

Hence Proved.

(ii) $\cos 100^\circ + \cos 20^\circ = \cos 40^\circ$

Let us consider LHS:

$$\cos 100^\circ + \cos 20^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned} \cos 100^\circ + \cos 20^\circ &= 2 \cos \frac{(100^\circ + 20^\circ)}{2} \cos \frac{(100^\circ - 20^\circ)}{2} \\ &= 2 \cos 120^\circ/2 \cos 80^\circ/2 \end{aligned}$$

$$\begin{aligned} &= 2 \cos 60^\circ \cos 4^\circ \\ &= 2 \times \frac{1}{2} \times \cos 40^\circ \\ &= \cos 40^\circ \\ &= \text{RHS} \end{aligned}$$

Hence Proved.

(iii) $\sin 50^\circ + \sin 10^\circ = \cos 20^\circ$

Let us consider LHS:

$$\sin 50^\circ + \sin 10^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned} \sin 50^\circ + \sin 10^\circ &= 2 \sin \frac{(50^\circ + 10^\circ)}{2} \cos \frac{(50^\circ - 10^\circ)}{2} \\ &= 2 \sin 60^\circ/2 \cos 40^\circ/2 \\ &= 2 \sin 30^\circ \cos 20^\circ \\ &= 2 \times \frac{1}{2} \times \cos 20^\circ \\ &= \cos 20^\circ \\ &= \text{RHS} \end{aligned}$$

Hence Proved.

(iv) $\sin 23^\circ + \sin 37^\circ = \cos 7^\circ$

Let us consider LHS:

$$\sin 23^\circ + \sin 37^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned} \sin 23^\circ + \sin 37^\circ &= 2 \sin \frac{(23^\circ + 37^\circ)}{2} \cos \frac{(23^\circ - 37^\circ)}{2} \\ &= 2 \sin 60^\circ/2 \cos -14^\circ/2 \\ &= 2 \sin 30^\circ \cos -7^\circ \\ &= 2 \times \frac{1}{2} \times \cos -7^\circ \\ &= \cos 7^\circ \text{ (since, } \{\cos (-A) = \cos A\}) \\ &= \text{RHS} \end{aligned}$$

Hence Proved.

(v) $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$

Let us consider LHS: $\sin 105^\circ + \cos 105^\circ$

$$\begin{aligned} \sin 105^\circ + \cos 105^\circ &= \sin 105^\circ + \sin (90^\circ - 105^\circ) \text{ [since, } \{\sin (90^\circ - A) = \cos A\}] \\ &= \sin 105^\circ + \sin (-15^\circ) \\ &= \sin 105^\circ - \sin 15^\circ \text{ [}\{\sin(-A) = -\sin A\}] \end{aligned}$$

By using the formula,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\begin{aligned}
 \sin 105^\circ - \sin 15^\circ &= 2 \cos (105^\circ + 15^\circ)/2 \sin (105^\circ - 15^\circ)/2 \\
 &= 2 \cos 120^\circ/2 \sin 90^\circ/2 \\
 &= 2 \cos 60^\circ \sin 45^\circ \\
 &= 2 \times 1/2 \times 1/\sqrt{2} \\
 &= 1/\sqrt{2} \\
 &= \cos 45^\circ \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

(vi) $\sin 40^\circ + \sin 20^\circ = \cos 10^\circ$

Let us consider LHS:

$$\sin 40^\circ + \sin 20^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

$$\begin{aligned}
 \sin 40^\circ + \sin 20^\circ &= 2 \sin (40^\circ + 20^\circ)/2 \cos (40^\circ - 20^\circ)/2 \\
 &= 2 \sin 60^\circ/2 \cos 20^\circ/2 \\
 &= 2 \sin 30^\circ \cos 10^\circ \\
 &= 2 \times 1/2 \times \cos 10^\circ \\
 &= \cos 10^\circ \\
 &= \text{RHS}
 \end{aligned}$$

Hence Proved.

3. Prove that:

(i) $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$

(ii) $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$

(iii) $\cos 80^\circ + \cos 40^\circ - \cos 20^\circ = 0$

(iv) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$

(v) $\sin 5\pi/18 - \cos 4\pi/9 = \sqrt{3} \sin \pi/9$

(vi) $\cos \pi/12 - \sin \pi/12 = 1/\sqrt{2}$

(vii) $\sin 80^\circ - \cos 70^\circ = \cos 50^\circ$

(viii) $\sin 51^\circ + \cos 81^\circ = \cos 21^\circ$

Solution:

(i) $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$

Let us consider LHS:

$$\cos 55^\circ + \cos 65^\circ + \cos 175^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\begin{aligned}
 \cos 55^\circ + \cos 65^\circ + \cos 175^\circ &= 2 \cos (55^\circ + 65^\circ)/2 \cos (55^\circ - 65^\circ) + \cos (180^\circ - 5^\circ) \\
 &= 2 \cos 120^\circ/2 \cos (-10^\circ)/2 - \cos 5^\circ \text{ (since, } \{\cos (180^\circ - A) = -\cos A\})
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \cos 60^\circ \cos (-5^\circ) - \cos 5^\circ \text{ (since, } \{\cos (-A) = \cos A\}) \\
 &= 2 \times 1/2 \times \cos 5^\circ - \cos 5^\circ \\
 &= 0 \\
 &= \text{RHS}
 \end{aligned}$$

Hence Proved.

(ii) $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$

Let us consider LHS:

$$\sin 50^\circ - \sin 70^\circ + \sin 10^\circ$$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\begin{aligned}
 \sin 50^\circ - \sin 70^\circ + \sin 10^\circ &= 2 \cos (50^\circ + 70^\circ)/2 \sin (50^\circ - 70^\circ) + \sin 10^\circ \\
 &= 2 \cos 120^\circ/2 \sin (-20^\circ)/2 + \sin 10^\circ \\
 &= 2 \cos 60^\circ (-\sin 10^\circ) + \sin 10^\circ \text{ [since, } \{\sin (-A) = -\sin (A)\}] \\
 &= 2 \times 1/2 \times -\sin 10^\circ + \sin 10^\circ \\
 &= 0 \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

(iii) $\cos 80^\circ + \cos 40^\circ - \cos 20^\circ = 0$

Let us consider LHS:

$$\cos 80^\circ + \cos 40^\circ - \cos 20^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\begin{aligned}
 \cos 80^\circ + \cos 40^\circ - \cos 20^\circ &= 2 \cos (80^\circ + 40^\circ)/2 \cos (80^\circ - 40^\circ) - \cos 20^\circ \\
 &= 2 \cos 120^\circ/2 \cos 40^\circ/2 - \cos 20^\circ \\
 &= 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ \\
 &= 2 \times 1/2 \times \cos 20^\circ - \cos 20^\circ \\
 &= 0 \\
 &= \text{RHS}
 \end{aligned}$$

Hence Proved.

(iv) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$

Let us consider LHS:

$$\cos 20^\circ + \cos 100^\circ + \cos 140^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\begin{aligned}
 \cos 20^\circ + \cos 100^\circ + \cos 140^\circ &= 2 \cos (20^\circ + 100^\circ)/2 \cos (20^\circ - 100^\circ) + \cos (180^\circ - 40^\circ) \\
 &= 2 \cos 120^\circ/2 \cos (-80^\circ)/2 - \cos 40^\circ \text{ (since, } \{\cos (180^\circ - A) = -\cos A\})
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \cos 60^\circ \cos (-40^\circ) - \cos 40^\circ \text{ (since, } \{\cos (-A) = \cos A\}) \\
 &= 2 \times 1/2 \times \cos 40^\circ - \cos 40^\circ \\
 &= 0 \\
 &= \text{RHS}
 \end{aligned}$$

Hence Proved.

$$(v) \sin 5\pi/18 - \cos 4\pi/9 = \sqrt{3} \sin \pi/9$$

Let us consider LHS:

$$\begin{aligned}
 \sin 5\pi/18 - \cos 4\pi/9 &= \sin 5\pi/18 - \sin (\pi/2 - 4\pi/9) \text{ (since, } \cos A = \sin (90^\circ - A)) \\
 &= \sin 5\pi/18 - \sin (9\pi - 8\pi)/18 \\
 &= \sin 5\pi/18 - \sin \pi/18
 \end{aligned}$$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\begin{aligned}
 \sin \frac{5\pi}{18} - \sin \frac{\pi}{18} &= 2 \cos \left(\frac{\frac{5\pi}{18} + \frac{\pi}{18}}{2} \right) \sin \left(\frac{\frac{5\pi}{18} - \frac{\pi}{18}}{2} \right) \\
 &= 2 \cos (6\pi/36) \sin (4\pi/36) \\
 &= 2 \cos \pi/6 \sin \pi/9 \\
 &= 2 \cos 30^\circ \sin \pi/9 \\
 &= 2 \times \sqrt{3}/2 \times \sin \pi/9 \\
 &= \sqrt{3} \sin \pi/9 \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

$$(vi) \cos \pi/12 - \sin \pi/12 = 1/\sqrt{2}$$

Let us consider LHS:

$$\begin{aligned}
 \cos \pi/12 - \sin \pi/12 &= \sin (\pi/2 - \pi/12) - \sin \pi/12 \text{ (since, } \cos A = \sin(90^\circ - A)) \\
 &= \sin (6\pi - 5\pi)/12 - \sin \pi/12 \\
 &= \sin 5\pi/12 - \sin \pi/12
 \end{aligned}$$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\begin{aligned}
 \sin \frac{5\pi}{12} - \sin \frac{\pi}{12} &= 2 \cos \left(\frac{\frac{5\pi}{12} + \frac{\pi}{12}}{2} \right) \sin \left(\frac{\frac{5\pi}{12} - \frac{\pi}{12}}{2} \right) \\
 &= 2 \cos (6\pi/24) \sin (4\pi/24) \\
 &= 2 \cos \pi/4 \sin \pi/6 \\
 &= 2 \cos 45^\circ \sin 30^\circ \\
 &= 2 \times 1/\sqrt{2} \times 1/2 \\
 &= 1/\sqrt{2} \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

$$(vii) \sin 80^\circ - \cos 70^\circ = \cos 50^\circ$$

$$\sin 80^\circ = \cos 50^\circ + \cos 70^\circ$$

So, now let us consider RHS

$$\cos 50^\circ + \cos 70^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos 50^\circ + \cos 70^\circ = 2 \cos \frac{(50^\circ + 70^\circ)}{2} \cos \frac{(50^\circ - 70^\circ)}{2}$$

$$= 2 \cos 120^\circ/2 \cos (-20^\circ)/2$$

$$= 2 \cos 60^\circ \cos (-10^\circ)$$

$$= 2 \times 1/2 \times \cos 10^\circ \text{ (since, } \cos(-A) = \cos A)$$

$$= \cos 10^\circ$$

$$= \cos (90^\circ - 80^\circ)$$

$$= \sin 80^\circ \text{ (since, } \cos(90^\circ - A) = \sin A)$$

$$= \text{LHS}$$

Hence Proved.

$$(viii) \sin 51^\circ + \cos 81^\circ = \cos 21^\circ$$

Let us consider LHS:

$$\sin 51^\circ + \cos 81^\circ = \sin 51^\circ + \sin (90^\circ - 81^\circ)$$

$$= \sin 51^\circ + \sin 9^\circ \text{ (since, } \sin(90^\circ - A) = \cos A)$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\sin 51^\circ + \sin 9^\circ = 2 \sin \frac{(51^\circ + 9^\circ)}{2} \cos \frac{(51^\circ - 9^\circ)}{2}$$

$$= 2 \sin 60^\circ/2 \cos 42^\circ/2$$

$$= 2 \sin 30^\circ \cos 21^\circ$$

$$= 2 \times 1/2 \times \cos 21^\circ$$

$$= \cos 21^\circ$$

$$= \text{RHS}$$

Hence proved.

4. Prove that:

$$(i) \cos \left(\frac{3\pi}{4} + x\right) - \cos \left(\frac{3\pi}{4} - x\right) = -\sqrt{2} \sin x$$

$$(ii) \cos \left(\frac{\pi}{4} + x\right) + \cos \left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x$$

Solution:

$$(i) \cos \left(\frac{3\pi}{4} + x\right) - \cos \left(\frac{3\pi}{4} - x\right) = -\sqrt{2} \sin x$$

Let us consider LHS:

$$\cos \left(\frac{3\pi}{4} + x\right) - \cos \left(\frac{3\pi}{4} - x\right)$$

By using the formula,

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\begin{aligned} \cos \left(\frac{3\pi}{4} + x \right) - \cos \left(\frac{3\pi}{4} - x \right) &= -2 \sin \frac{\left(\frac{3\pi}{4} + x + \frac{3\pi}{4} - x \right)}{2} \sin \frac{\left(\frac{3\pi}{4} + x - \frac{3\pi}{4} + x \right)}{2} \\ &= -2 \sin \frac{(6\pi/4)}{2} \sin \frac{2x}{2} \\ &= -2 \sin \frac{6\pi}{8} \sin x \\ &= -2 \sin \frac{3\pi}{4} \sin x \\ &= -2 \sin \left(\pi - \frac{\pi}{4} \right) \sin x \\ &= -2 \sin \frac{\pi}{4} \sin x \text{ (since, } (\pi - A) = \sin A) \\ &= -2 \times \frac{1}{\sqrt{2}} \times \sin x \\ &= -\sqrt{2} \sin x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

(ii) $\cos \left(\frac{\pi}{4} + x \right) + \cos \left(\frac{\pi}{4} - x \right) = \sqrt{2} \cos x$

Let us consider LHS:

$$\cos \left(\frac{\pi}{4} + x \right) + \cos \left(\frac{\pi}{4} - x \right)$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned} \cos \left(\frac{\pi}{4} + x \right) + \cos \left(\frac{\pi}{4} - x \right) &= 2 \cos \frac{\left(\frac{\pi}{4} + x + \frac{\pi}{4} - x \right)}{2} \cos \frac{\left(\frac{\pi}{4} + x - \frac{\pi}{4} + x \right)}{2} \\ &= 2 \cos \frac{(2\pi/4)}{2} \cos \frac{2x}{2} \\ &= 2 \cos \frac{2\pi}{8} \cos x \\ &= 2 \sin \frac{\pi}{4} \cos x \\ &= 2 \times \frac{1}{\sqrt{2}} \times \cos x \\ &= \sqrt{2} \cos x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

5. Prove that:

(i) $\sin 65^\circ + \cos 65^\circ = \sqrt{2} \cos 20^\circ$

(ii) $\sin 47^\circ + \cos 77^\circ = \cos 17^\circ$

Solution:

(i) $\sin 65^\circ + \cos 65^\circ = \sqrt{2} \cos 20^\circ$

Let us consider LHS:

$$\begin{aligned} \sin 65^\circ + \cos 65^\circ &= \sin 65^\circ + \sin (90^\circ - 65^\circ) \\ &= \sin 65^\circ + \sin 25^\circ \end{aligned}$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned} \sin 65^\circ + \sin 25^\circ &= 2 \sin \frac{(65^\circ + 25^\circ)}{2} \cos \frac{(65^\circ - 25^\circ)}{2} \\ &= 2 \sin 90^\circ/2 \cos 40^\circ/2 \end{aligned}$$

$$\begin{aligned}
 &= 2 \sin 45^\circ \cos 20^\circ \\
 &= 2 \times \frac{1}{\sqrt{2}} \times \cos 20^\circ \\
 &= \sqrt{2} \cos 20^\circ \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

(ii) $\sin 47^\circ + \cos 77^\circ = \cos 17^\circ$

Let us consider LHS:

$$\begin{aligned}
 \sin 47^\circ + \cos 77^\circ &= \sin 47^\circ + \sin (90^\circ - 77^\circ) \\
 &= \sin 47^\circ + \sin 13^\circ
 \end{aligned}$$

By using the formula,

$$\begin{aligned}
 \sin A + \sin B &= 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2} \\
 \sin 47^\circ + \sin 13^\circ &= 2 \sin \frac{(47^\circ + 13^\circ)}{2} \cos \frac{(47^\circ - 13^\circ)}{2} \\
 &= 2 \sin 60^\circ/2 \cos 34^\circ/2 \\
 &= 2 \sin 30^\circ \cos 17^\circ \\
 &= 2 \times \frac{1}{2} \times \cos 17^\circ \\
 &= \cos 17^\circ \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

6. Prove that:

- (i)** $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A$
- (ii)** $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A$
- (iii)** $\sin A + \sin 2A + \sin 4A + \sin 5A = 4 \cos A/2 \cos 3A/2 \sin 3A$
- (iv)** $\sin 3A + \sin 2A - \sin A = 4 \sin A \cos A/2 \cos 3A/2$
- (v)** $\cos 20^\circ \cos 100^\circ + \cos 100^\circ \cos 140^\circ - \cos 140^\circ \cos 200^\circ = -3/4$
- (vi)** $\sin x/2 \sin 7x/2 + \sin 3x/2 \sin 11x/2 = \sin 2x \sin 5x$
- (vii)** $\cos x \cos x/2 - \cos 3x \cos 9x/2 = \sin 4x \sin 7x/2$

Solution:

(i) $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A$

Let us consider LHS:

$$\cos 3A + \cos 5A + \cos 7A + \cos 15A$$

So now,

$$(\cos 5A + \cos 3A) + (\cos 15A + \cos 7A)$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$(\cos 5A + \cos 3A) + (\cos 15A + \cos 7A)$$

$$= [2 \cos \frac{(5A+3A)}{2} \cos \frac{(5A-3A)}{2}] + [2 \cos \frac{(15A+7A)}{2} \cos \frac{(15A-7A)}{2}]$$

$$= [2 \cos 8A/2 \cos 2A/2] + [2 \cos 22A/2 \cos 8A/2]$$

$$= [2 \cos 4A \cos A] + [2 \cos 11A \cos 4A]$$

$$= 2 \cos 4A (\cos 11A + \cos A)$$

Again by using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$2 \cos 4A (\cos 11A + \cos A) = 2 \cos 4A [2 \cos \frac{(11A+A)}{2} \cos \frac{(11A-A)}{2}]$$

$$= 2 \cos 4A [2 \cos 12A/2 \cos 10A/2]$$

$$= 2 \cos 4A [2 \cos 6A \cos 5A]$$

$$= 4 \cos 4A \cos 5A \cos 6A$$

$$= \text{RHS}$$

Hence proved.

(ii) $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A$

Let us consider LHS:

$$\cos A + \cos 3A + \cos 5A + \cos 7A$$

So now,

$$(\cos 3A + \cos A) + (\cos 7A + \cos 5A)$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$(\cos 3A + \cos A) + (\cos 7A + \cos 5A)$$

$$= [2 \cos \frac{(3A+A)}{2} \cos \frac{(3A-A)}{2}] + [2 \cos \frac{(7A+5A)}{2} \cos \frac{(7A-5A)}{2}]$$

$$= [2 \cos 4A/2 \cos 2A/2] + [2 \cos 12A/2 \cos 2A/2]$$

$$= [2 \cos 2A \cos A] + [2 \cos 6A \cos A]$$

$$= 2 \cos A (\cos 6A + \cos 2A)$$

Again by using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$2 \cos A (\cos 6A + \cos 2A) = 2 \cos A [2 \cos \frac{(6A+2A)}{2} \cos \frac{(6A-2A)}{2}]$$

$$= 2 \cos A [2 \cos 8A/2 \cos 4A/2]$$

$$= 2 \cos A [2 \cos 4A \cos 2A]$$

$$= 4 \cos A \cos 2A \cos 4A$$

$$= \text{RHS}$$

Hence proved.

(iii) $\sin A + \sin 2A + \sin 4A + \sin 5A = 4 \cos A/2 \cos 3A/2 \sin 3A$

Let us consider LHS:

$$\sin A + \sin 2A + \sin 4A + \sin 5A$$

So now,

$$(\sin 2A + \sin A) + (\sin 5A + \sin 4A)$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned}
 (\sin 2A + \sin A) + (\sin 5A + \sin 4A) &= \\
 &= [2 \sin (2A+A)/2 \cos (2A-A)/2] + [2 \sin (5A+4A)/2 \cos (5A-4A)/2] \\
 &= [2 \sin 3A/2 \cos A/2] + [2 \sin 9A/2 \cos A/2] \\
 &= 2 \cos A/2 (\sin 9A/2 + \sin 3A/2)
 \end{aligned}$$

Again by using the formula,

$$\begin{aligned}
 \sin A + \sin B &= 2 \sin (A+B)/2 \cos (A-B)/2 \\
 2 \cos A/2 (\sin 9A/2 + \sin 3A/2) &= 2 \cos A/2 [2 \sin (9A/2 + 3A/2)/2 \cos (9A/2 - 3A/2)/2] \\
 &= 2 \cos A/2 [2 \sin ((9A+3A)/2)/2 \cos ((9A-3A)/2)/2] \\
 &= 2 \cos A/2 [2 \sin 12A/4 \cos 6A/4] \\
 &= 2 \cos A/2 [2 \sin 3A \cos 3A/2] \\
 &= 4 \cos A/2 \cos 3A/2 \sin 3A \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

(iv) $\sin 3A + \sin 2A - \sin A = 4 \sin A \cos A/2 \cos 3A/2$

Let us consider LHS:

$$\sin 3A + \sin 2A - \sin A$$

So now,

$$(\sin 3A - \sin A) + \sin 2A$$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\begin{aligned}
 (\sin 3A - \sin A) + \sin 2A &= 2 \cos (3A + A)/2 \sin (3A - A)/2 + \sin 2A \\
 &= 2 \cos 4A/2 \sin 2A/2 + \sin 2A \\
 \text{We know that, } \sin 2A &= 2 \sin A \cos A \\
 &= 2 \cos 2A \sin A + 2 \sin A \cos A \\
 &= 2 \sin A (\cos 2A + \cos A)
 \end{aligned}$$

Again by using the formula,

$$\begin{aligned}
 \cos A + \cos B &= 2 \cos (A+B)/2 \cos (A-B)/2 \\
 2 \sin A (\cos 2A + \cos A) &= 2 \sin A [2 \cos (2A+A)/2 \cos (2A-A)/2] \\
 &= 2 \sin A [2 \cos 3A/2 \cos A/2] \\
 &= 4 \sin A \cos A/2 \cos 3A/2 \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

(v) $\cos 20^\circ \cos 100^\circ + \cos 100^\circ \cos 140^\circ - \cos 140^\circ \cos 200^\circ = -3/4$

Let us consider LHS:

$$\cos 20^\circ \cos 100^\circ + \cos 100^\circ \cos 140^\circ - \cos 140^\circ \cos 200^\circ =$$

We shall multiply and divide by 2 we get,

$$= 1/2 [2 \cos 100^\circ \cos 20^\circ + 2 \cos 140^\circ \cos 100^\circ - 2 \cos 200^\circ \cos 140^\circ]$$

We know that $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

So,

$$\begin{aligned} &= 1/2 [\cos (100^\circ + 20^\circ) + \cos (100^\circ - 20^\circ) + \cos (140^\circ + 100^\circ) + \cos (140^\circ - 100^\circ) - \cos (200^\circ + 140^\circ) - \cos (200^\circ - 140^\circ)] \\ &= 1/2 [\cos 120^\circ + \cos 80^\circ + \cos 240^\circ + \cos 40^\circ - \cos 340^\circ - \cos 60^\circ] \\ &= 1/2 [\cos (90^\circ + 30^\circ) + \cos 80^\circ + \cos (180^\circ + 60^\circ) + \cos 40^\circ - \cos (360^\circ - 20^\circ) - \cos 60^\circ] \end{aligned}$$

We know, $\cos (180^\circ + A) = -\cos A$, $\cos (90^\circ + A) = -\sin A$, $\cos (360^\circ - A) = \cos A$

So,

$$\begin{aligned} &= 1/2 [-\sin 30^\circ + \cos 80^\circ - \cos 60^\circ + \cos 40^\circ - \cos 20^\circ - \cos 60^\circ] \\ &= 1/2 [-\sin 30^\circ + \cos 80^\circ + \cos 40^\circ - \cos 20^\circ - 2 \cos 60^\circ] \end{aligned}$$

Again by using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\begin{aligned} &= 1/2 [-\sin 30^\circ + 2 \cos (80^\circ+40^\circ)/2 \cos (80^\circ-40^\circ)/2 - \cos 20^\circ - 2 \times 1/2] \\ &= 1/2 [-\sin 30^\circ + 2 \cos 120^\circ/2 \cos 40^\circ/2 - \cos 20^\circ - 1] \\ &= 1/2 [-\sin 30^\circ + 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ - 1] \\ &= 1/2 [-1/2 + 2 \times 1/2 \times \cos 20^\circ - \cos 20^\circ - 1] \\ &= 1/2 [-1/2 + \cos 20^\circ - \cos 20^\circ - 1] \\ &= 1/2 [-1/2 - 1] \\ &= 1/2 [-3/2] \\ &= -3/4 \\ &= \text{RHS} \end{aligned}$$

Hence proved.

(vi) $\sin x/2 \sin 7x/2 + \sin 3x/2 \sin 11x/2 = \sin 2x \sin 5x$

Let us consider LHS:

$$\sin x/2 \sin 7x/2 + \sin 3x/2 \sin 11x/2 =$$

We shall multiply and divide by 2 we get,

$$= 1/2 [2 \sin 7x/2 \sin x/2 + 2 \sin 11x/2 \sin 3x/2]$$

We know that $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$

So,

$$\begin{aligned} &= 1/2 [\cos (7x/2 - x/2) - \cos (7x/2 + x/2) + \cos (11x/2 - 3x/2) - \cos (11x/2 + 3x/2)] \\ &= 1/2 [\cos (7x-x)/2 - \cos (7x+x)/2 + \cos (11x-3x)/2 - \cos (11x+3x)/2] \\ &= 1/2 [\cos 6x/2 - \cos 8x/2 + \cos 8x/2 - \cos 14x/2] \\ &= 1/2 [\cos 3x - \cos 7x] \\ &= -1/2 [\cos 7x - \cos 3x] \end{aligned}$$

Again by using the formula,

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

$$= -1/2 [-2 \sin (7x+3x)/2 \sin (7x-3x)/2]$$

$$\begin{aligned}
 &= -1/2 [-2 \sin 10x/2 \sin 4x/2] \\
 &= -1/2 [-2 \sin 5x \sin 2x] \\
 &= -2/-2 \sin 5x \sin 2x \\
 &= \sin 2x \sin 5x \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

(vii) $\cos x \cos x/2 - \cos 3x \cos 9x/2 = \sin 4x \sin 7x/2$

Let us consider LHS:

$$\cos x \cos x/2 - \cos 3x \cos 9x/2 =$$

We shall multiply and divide by 2 we get,

$$= 1/2 [2 \cos x \cos x/2 - 2 \cos 9x/2 \cos 3x]$$

We know that $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

So,

$$\begin{aligned}
 &= 1/2 [\cos (x + x/2) + \cos (x - x/2) - \cos (9x/2 + 3x) - \cos (9x/2 - 3x)] \\
 &= 1/2 [\cos (2x+x)/2 + \cos (2x-x)/2 - \cos (9x+6x)/2 - \cos (9x-6x)/2] \\
 &= 1/2 [\cos 3x/2 + \cos x/2 - \cos 15x/2 - \cos 3x/2] \\
 &= 1/2 [\cos x/2 - \cos 15x/2] \\
 &= -1/2 [\cos 15x/2 - \cos x/2]
 \end{aligned}$$

Again by using the formula,

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

$$\begin{aligned}
 &= -1/2 [-2 \sin (15x/2 + x/2)/2 \sin (15x/2 - x/2)/2] \\
 &= -1/2 [-2 \sin (16x/2)/2 \sin (14x/2)/2] \\
 &= -1/2 [-2 \sin 16x/4 \sin 7x/2] \\
 &= -1/2 [-2 \sin 4x \sin 7x/2] \\
 &= -2/-2 [\sin 4x \sin 7x/2] \\
 &= \sin 4x \sin 7x/2 \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

7. Prove that:

(i) $\frac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$

(ii) $\frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} = \cot 8A$

(iii) $\frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A - B}{2}$

$$(iv) \frac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$$

$$(v) \frac{\cos A + \cos B}{\cos B - \cos A} = \cot\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$$

Solution:

$$(i) \frac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$$

Solution:

Let us consider LHS:

$$\frac{\sin A + \sin 3A}{\cos A - \cos 3A}$$

By using the formulas,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

So now,

$$\begin{aligned} \frac{\sin 3A + \sin A}{\cos A - \cos 3A} &= \frac{2\left(\sin \frac{A+3A}{2} \cos \frac{3A-A}{2}\right)}{-2\left(\sin \frac{A+3A}{2} \sin \frac{A-3A}{2}\right)} \\ &= -\frac{\sin \frac{4A}{2} \cos \frac{2A}{2}}{\sin \frac{4A}{2} \sin \frac{-2A}{2}} \\ &= -\frac{\cos A}{-\sin A} \quad (\text{since, } \sin(-A) = -\sin A) \\ &= \frac{\cos A}{\sin A} \\ &= \cot A \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(ii) \frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} = \cot 8A$$

Solution:

Let us consider LHS:

$$\frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

So now,

$$\begin{aligned} \frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} &= \frac{2(\cos \frac{9A+7A}{2} \sin \frac{9A-7A}{2})}{-2(\sin \frac{7A+9A}{2} \sin \frac{7A-9A}{2})} \\ &= -\frac{(\cos \frac{16A}{2} \sin \frac{2A}{2})}{(\sin \frac{16A}{2} \sin \frac{(-2A)}{2})} \\ &= -\frac{(\cos 8A \sin A)}{(-\sin 8A \sin A)} \text{ (since, } \sin(-A) = -\sin A) \\ &= \frac{\cos 8A}{\sin 8A} \\ &= \cot A \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(iii) \frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A-B}{2}$$

Solution:

Let us consider LHS:

$$\frac{\sin A - \sin B}{\cos A + \cos B}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned} \frac{\sin A - \sin B}{\cos A + \cos B} &= \frac{2(\cos \frac{A+B}{2} \sin \frac{A-B}{2})}{2(\cos \frac{A+B}{2} \cos \frac{A-B}{2})} \\ &= \frac{\sin(\frac{A-B}{2})}{\cos(\frac{A-B}{2})} \\ &= \tan \frac{A-B}{2} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(iv) \frac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$$

Solution:

Let us consider LHS:

$$\frac{\sin A + \sin B}{\sin A - \sin B}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\left(\sin \frac{A+B}{2} \cos \frac{A-B}{2}\right)}{2\left(\cos \frac{A+B}{2} \sin \frac{A-B}{2}\right)}$$

$$= \frac{\sin \frac{A+B}{2} \cos \frac{A-B}{2}}{\cos \frac{A+B}{2} \sin \frac{A-B}{2}}$$

$$= \tan\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$$

$$= \text{RHS}$$

Hence proved.

$$(v) \frac{\cos A + \cos B}{\cos B - \cos A} = \cot\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$$

Solution:

Let us consider LHS:

$$\frac{\cos A + \cos B}{\cos B - \cos A}$$

By using the formulas,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

So now,

$$\frac{\cos A + \cos B}{\cos B - \cos A} = \frac{2\left(\cos \frac{A+B}{2} \cos \frac{A-B}{2}\right)}{-2\left(\sin \frac{A+B}{2} \sin \frac{B-A}{2}\right)}$$

$$= -\frac{\cos \frac{A+B}{2} \cos \frac{A-B}{2}}{\sin \frac{A+B}{2} \sin\left(-\frac{A-B}{2}\right)}$$

$$\begin{aligned}
 &= -\frac{\cos \frac{A+B}{2} \cos \frac{A-B}{2}}{-\sin \frac{A+B}{2} \sin \frac{A-B}{2}} \text{ (since, } \sin(-x) = -\sin x \text{)} \\
 &= \cot\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right) \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

8. Prove that:

$$(i) \frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$$

$$(ii) \frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{\cos 5A}{\cos 3A}$$

$$(iii) \frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$$

$$(iv) \frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$$

$$(v) \frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$$

$$(vi) \frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$$

$$(vii) \frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$$

$$(viii) \frac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \tan 2A$$

$$(ix) \frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$$

$$(x) \frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}$$

$$(xi) \frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} = \tan\theta$$

Solution:

$$(i) \frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$$

Let us consider LHS:

$$\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A}$$

By using the formulas,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \frac{(\sin 5A + \sin A) + \sin 3A}{(\cos 5A + \cos A) + \cos 3A}$$

$$= \frac{\left(2 \sin \frac{5A+A}{2} \cos \frac{5A-A}{2}\right) + \sin 3A}{\left(2 \cos \frac{5A+A}{2} \cos \frac{5A-A}{2}\right) + \cos 3A}$$

$$= \frac{\left(2 \sin \frac{6A}{2} \cos \frac{4A}{2}\right) + \sin 3A}{\left(2 \cos \frac{6A}{2} \cos \frac{4A}{2}\right) + \cos 3A}$$

$$= \frac{(2 \sin 3A \cos 2A) + \sin 3A}{(2 \cos 3A \cos 2A) + \cos 3A}$$

$$= \frac{\sin 3A (2 \cos 2A + 1)}{\cos 3A (2 \cos 2A + 1)}$$

$$= \tan 3A$$

$$= \text{RHS}$$

Hence proved.

$$(ii) \frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{\cos 5A}{\cos 3A}$$

Let us consider LHS:

$$\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A}$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned} \frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} &= \frac{(\cos 7A + \cos 3A) + 2\cos 5A}{(\cos 5A + \cos A) + 2\cos 3A} \\ &= \frac{\left(2\cos \frac{7A+3A}{2} \cos \frac{7A-3A}{2}\right) + 2\cos 5A}{\left(2\cos \frac{5A+A}{2} \cos \frac{5A-A}{2}\right) + 2\cos 3A} \\ &= \frac{\left(2\cos \frac{10A}{2} \cos \frac{4A}{2}\right) + 2\cos 5A}{\left(2\cos \frac{6A}{2} \cos \frac{4A}{2}\right) + 2\cos 3A} \\ &= \frac{(2\cos 5A \cos 2A) + 2\cos 5A}{(2\cos 3A \cos 2A) + 2\cos 3A} \\ &= \frac{2\cos 5A(\cos 2A + 1)}{2\cos 3A(\cos 2A + 1)} \\ &= \frac{\cos 5A}{\cos 3A} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$\text{(iii)} \frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$$

Let us consider LHS:

$$\frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A}$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \frac{(\cos 4A + \cos 2A) + \cos 3A}{(\sin 4A + \sin 2A) + \sin 3A}$$

$$\begin{aligned}
 &= \frac{\left(2 \cos \frac{4A + 2A}{2} \cos \frac{4A - 2A}{2}\right) + \cos 3A}{\left(2 \sin \frac{4A + 2A}{2} \cos \frac{4A - 2A}{2}\right) + \sin 3A} \\
 &= \frac{\left(2 \cos \frac{6A}{2} \cos \frac{2A}{2}\right) + \cos 3A}{\left(2 \sin \frac{6A}{2} \cos \frac{2A}{2}\right) + \sin 3A} \\
 &= \frac{(2 \cos 3A \cos A) + \cos 3A}{(2 \sin 3A \cos A) + \sin 3A} \\
 &= \frac{\cos 3A (2 \cos A + 1)}{\sin 3A (2 \cos A + 1)} \\
 &= \cot 3A \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

$$(iv) \frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$$

Let us consider LHS:

$$\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A}$$

By using the formulas,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A}$$

When we rearrange we get,

$$\begin{aligned}
 &= \frac{(\sin 9A + \sin 3A) + (\sin 7A + \sin 5A)}{(\cos 9A + \cos 3A) + (\cos 7A + \cos 5A)} \\
 &= \frac{\left(2 \sin \frac{9A + 3A}{2} \cos \frac{9A - 3A}{2}\right) + \left(2 \sin \frac{7A + 5A}{2} \cos \frac{7A - 5A}{2}\right)}{\left(2 \cos \frac{9A + 3A}{2} \cos \frac{9A - 3A}{2}\right) + \left(2 \cos \frac{7A + 5A}{2} \cos \frac{7A - 5A}{2}\right)} \\
 &= \frac{\left(2 \sin \frac{12A}{2} \cos \frac{6A}{2}\right) + \left(2 \sin \frac{12A}{2} \cos \frac{2A}{2}\right)}{\left(2 \cos \frac{12A}{2} \cos \frac{6A}{2}\right) + \left(2 \cos \frac{12A}{2} \cos \frac{2A}{2}\right)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(2 \sin 6A \cos 3A) + (2 \sin 6A \cos A)}{(2 \cos 6A \cos 3A) + (2 \cos 6A \cos A)} \\
 &= \frac{2 \sin 6A (\cos 3A + \cos A)}{2 \cos 6A (\cos 3A + \cos A)} \\
 &= \frac{\sin 6A}{\cos 6A} \\
 &= \tan 6A \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

$$(v) \frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$$

Let us consider LHS:

$$\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A}$$

By using the formulas,

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

So now,

$$\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \frac{-(\sin 7A - \sin 5A) + (\sin 8A - \sin 4A)}{(\cos 7A - \cos 5A) - (\cos 8A - \cos 4A)}$$

When we rearrange we get,

$$\begin{aligned}
 &= \frac{-\left(2 \cos \frac{7A + 5A}{2} \sin \frac{7A - 5A}{2}\right) + \left(2 \cos \frac{8A + 4A}{2} \sin \frac{8A - 4A}{2}\right)}{\left(-2 \sin \frac{7A + 5A}{2} \sin \frac{7A - 5A}{2}\right) - \left(-2 \sin \frac{8A + 4A}{2} \sin \frac{8A - 4A}{2}\right)} \\
 &= \frac{-\left(2 \cos \frac{12A}{2} \sin \frac{2A}{2}\right) + \left(2 \cos \frac{12A}{2} \sin \frac{4A}{2}\right)}{\left(-2 \sin \frac{12A}{2} \sin \frac{2A}{2}\right) - \left(-2 \sin \frac{12A}{2} \sin \frac{4A}{2}\right)} \\
 &= \frac{-(2 \cos 6A \sin A) + (2 \cos 6A \sin 2A)}{-(2 \sin 6A \sin A) + (2 \sin 6A \sin 2A)} \\
 &= \frac{2 \cos 6A (-\sin A + \sin 2A)}{2 \sin 6A (-\sin A + \sin 2A)} \\
 &= \frac{\cos 6A}{\sin 6A}
 \end{aligned}$$

$$= \cot 6A$$

$$= \text{RHS}$$

Hence proved.

$$(vi) \frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$$

Let us consider LHS:

$$\frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \frac{(2 \sin 5A \cos 2A) - (2 \sin 6A \cos A)}{(2 \sin 2A \sin A) - (2 \cos 3A \cos 3A)}$$

We know that, $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$,

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$ and

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

So now,

$$= \frac{\{\sin(5A + 2A) + \sin(5A - 2A)\} - \{\sin(6A + A) + \sin(6A - A)\}}{\{\cos(2A - A) - \cos(2A + A)\} - \{\cos(3A + 2A) + \cos(3A - 2A)\}}$$

$$= \frac{\{\sin 7A + \sin 3A\} - \{\sin 7A + \sin 5A\}}{\{\cos A - \cos 3A\} - \{\cos 5A + \cos A\}}$$

$$= \frac{\sin 7A + \sin 3A - \sin 7A - \sin 5A}{\cos A - \cos 3A - \cos 5A - \cos A}$$

$$= \frac{\sin 3A - \sin 5A}{-(\cos 5A + \cos 3A)}$$

$$= \frac{-(\sin 5A - \sin 3A)}{-(\cos 5A + \cos 3A)}$$

$$= \frac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

so,

$$\begin{aligned}
 &= \frac{2 \cos \frac{5A + 3A}{2} \sin \frac{5A - 3A}{2}}{2 \cos \frac{5A + 3A}{2} \cos \frac{5A - 3A}{2}} \\
 &= \frac{2 \cos \frac{8A}{2} \sin \frac{2A}{2}}{2 \cos \frac{8A}{2} \cos \frac{2A}{2}} \\
 &= \frac{\sin A}{\cos A} \\
 &= \tan A \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

$$(vii) \frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$$

Let us consider LHS:

$$\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \frac{(2 \sin 11A \sin A) + (2 \sin 7A \sin 3A)}{(2 \cos 11A \sin A) + (2 \cos 7A \sin 3A)}$$

We know that, $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$,

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

So now,

$$\begin{aligned}
 &= \frac{\{\cos(11A - A) - \cos(11A + A)\} + \{\cos(7A - 3A) - \cos(7A + 3A)\}}{\{\sin(11A + A) - \sin(11A - A)\} + \{\sin(7A + 3A) - \sin(7A - 3A)\}} \\
 &= \frac{\{\cos 10A - \cos 12A\} + \{\cos 4A - \cos 10A\}}{\{\sin 12A - \sin 10A\} + \{\sin 10A - \sin 4A\}} \\
 &= \frac{\cos 10A - \cos 12A + \cos 4A - \cos 10A}{\sin 12A - \sin 10A + \sin 10A - \sin 4A} \\
 &= \frac{-(\cos 12A - \cos 4A)}{\sin 12A - \sin 4A}
 \end{aligned}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

so,

$$\begin{aligned}
 &= -\frac{-2 \sin \frac{12A + 4A}{2} \sin \frac{12A - 4A}{2}}{2 \cos \frac{12A + 4A}{2} \sin \frac{12A - 4A}{2}} \\
 &= \frac{\sin \frac{16A}{2} \sin \frac{8A}{2}}{\cos \frac{16A}{2} \sin \frac{8A}{2}} \\
 &= \frac{\sin 8A}{\cos 8A} \\
 &= \tan 8A \\
 &= \text{RHS} \\
 &\text{Hence proved.}
 \end{aligned}$$

$$(viii) \frac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \tan 2A$$

Let us consider LHS:

$$\frac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \frac{(2 \sin 3A \cos 4A) - (2 \sin A \cos 2A)}{(2 \sin 4A \sin A) + (2 \cos 6A \cos A)}$$

We know that, $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$,

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

So now,

$$\begin{aligned}
 &= \frac{\{\sin(3A + 4A) + \sin(3A - 4A)\} - \{\sin(A + 2A) + \sin(A - 2A)\}}{\{\cos(4A - A) - \cos(4A + A)\} + \{\cos(6A + A) + \cos(6A - A)\}} \\
 &= \frac{\{\sin 7A + \sin(-A)\} - \{\sin 3A + \sin(-A)\}}{\{\cos 3A - \cos 5A\} + \{\cos 7A + \cos 5A\}} \\
 &= \frac{\{\sin 7A - \sin A\} - \{\sin 3A - \sin A\}}{\{\cos 3A - \cos 5A\} + \{\cos 7A + \cos 5A\}} \quad (\text{since, } (-A) = -\sin A) \\
 &= \frac{\sin 7A - \sin A - \sin 3A + \sin A}{\cos 3A - \cos 5A + \cos 7A + \cos 5A}
 \end{aligned}$$

$$= \frac{\sin 7A - \sin 3A}{\cos 7A + \cos 3A}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

so,

$$= \frac{2 \cos \frac{7A + 3A}{2} \sin \frac{7A - 3A}{2}}{2 \cos \frac{7A + 3A}{2} \cos \frac{7A - 3A}{2}}$$

$$= \frac{\cos \frac{10A}{2} \sin \frac{4A}{2}}{\cos \frac{10A}{2} \cos \frac{4A}{2}}$$

$$= \frac{\sin 2A}{\cos 2A}$$

$$= \tan 2A$$

$$= \text{RHS}$$

Hence proved.

$$(ix) \frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$$

Let us consider LHS:

$$\frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \frac{(2 \sin 2A \sin A) + (2 \sin 6A \sin 3A)}{(2 \sin A \cos 2A) + (2 \sin 3A \cos 6A)}$$

We know that, $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$,

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

So now,

$$\begin{aligned} &= \frac{\{\cos(2A - A) - \cos(2A + A)\} + \{\cos(6A - 3A) - \cos(6A + 3A)\}}{\{\sin(A + 2A) + \sin(A - 2A)\} + \{\sin(3A + 6A) + \sin(3A - 6A)\}} \\ &= \frac{\{\cos A - \cos 3A\} + \{\cos 3A - \cos 9A\}}{\{\sin 3A + \sin(-A)\} + \{\sin 9A + \sin(-3A)\}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\{\cos A - \cos 3A\} + \{\cos 3A - \cos 9A\}}{\{\sin 3A - \sin A\} + \{\sin 9A - \sin 3A\}} \quad (\text{since, } \sin(-A) = -\sin A) \\
 &= \frac{\cos A - \cos 3A + \cos 3A - \cos 9A}{\sin 3A - \sin A + \sin 9A - \sin 3A} \\
 &= \frac{\cos A - \cos 9A}{\sin 9A - \sin A} \\
 &= \frac{-(\cos 9A - \cos A)}{\sin 9A - \sin A}
 \end{aligned}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

so,

$$= -\frac{-2 \sin \frac{9A + A}{2} \sin \frac{9A - A}{2}}{2 \cos \frac{9A + A}{2} \sin \frac{9A - A}{2}}$$

$$= \frac{\sin \frac{10A}{2} \sin \frac{8A}{2}}{\cos \frac{10A}{2} \sin \frac{8A}{2}}$$

$$= \frac{\sin 5A}{\cos 5A}$$

$$= \tan 5A$$

$$= \text{RHS}$$

Hence proved.

$$(x) \frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}$$

Let us consider LHS:

$$\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A}$$

$$\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{(\sin 5A + \sin A) + 2\sin 3A}{(\sin 7A + \sin 3A) + 2\sin 5A}$$

We know that,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned}
 &= \frac{\left(2 \sin \frac{5A + A}{2} \cos \frac{5A - A}{2}\right) + 2 \sin 3A}{\left(2 \sin \frac{7A + 3A}{2} \cos \frac{7A - 3A}{2}\right) + 2 \sin 5A} \\
 &= \frac{\left(2 \sin \frac{6A}{2} \cos \frac{4A}{2}\right) + 2 \sin 3A}{\left(2 \sin \frac{10A}{2} \cos \frac{4A}{2}\right) + 2 \sin 5A} \\
 &= \frac{(2 \sin 3A \cos 2A) + 2 \sin 3A}{(2 \sin 5A \cos 2A) + 2 \sin 5A} \\
 &= \frac{2 \sin 3A (\cos 2A + 1)}{2 \sin 5A (\cos 2A + 1)} \\
 &= \frac{\sin 3A}{\sin 5A} \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

(xi) $\frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} = \tan\theta$

Let us consider LHS:

$$\begin{aligned}
 &\frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} \\
 &\frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} = \frac{\{\sin(\theta + \phi) + \sin(\theta - \phi)\} - 2\sin\theta}{\{\cos(\theta + \phi) + \cos(\theta - \phi)\} - 2\cos\theta}
 \end{aligned}$$

We know that,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned}
 &= \frac{\left(2 \sin \frac{\theta + \phi + \theta - \phi}{2} \cos \frac{\theta + \phi - \theta + \phi}{2}\right) - 2 \sin \theta}{\left(2 \cos \frac{\theta + \phi + \theta - \phi}{2} \cos \frac{\theta + \phi - \theta + \phi}{2}\right) - 2 \cos \theta} \\
 &= \frac{\left(2 \sin \frac{2\theta}{2} \cos \frac{2\phi}{2}\right) - 2 \sin \theta}{\left(2 \cos \frac{2\theta}{2} \cos \frac{2\phi}{2}\right) - 2 \cos \theta}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(2 \sin \theta \cos \Phi) - 2 \sin \theta}{(2 \cos \theta \cos \Phi) - 2 \cos \theta} \\
 &= \frac{2 \sin \theta (\cos \Phi - 1)}{2 \cos \theta (\cos \Phi - 1)} \\
 &= \frac{\sin \theta}{\cos \theta} \\
 &= \tan \theta \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

9. Prove that:

(i) $\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) = 4 \sin (\alpha + \beta)/2 \sin (\beta + \gamma)/2 \sin (\alpha + \gamma)/2$

(ii) $\cos (A + B + C) + \cos (A - B + C) + \cos (A + B - C) + \cos (-A + B + C) = 4 \cos A \cos B \cos C$

Solution:

(i) $\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) = 4 \sin (\alpha + \beta)/2 \sin (\beta + \gamma)/2 \sin (\alpha + \gamma)/2$

Let us consider LHS:

$$\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma)$$

By using the formulas,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\begin{aligned}
 &\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) \\
 &= (\sin \alpha + \sin \beta) + \{\sin \gamma - \sin (\alpha + \beta + \gamma)\} \\
 &= \left(2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}\right) + \left(2 \cos \frac{\gamma + \alpha + \beta + \gamma}{2} \sin \frac{\gamma - \alpha - \beta - \gamma}{2}\right) \\
 &= \left(2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}\right) + \left(2 \cos \frac{\alpha + \beta + 2\gamma}{2} \sin \frac{-(\alpha + \beta)}{2}\right) \\
 &= \left(2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}\right) - \left(2 \cos \frac{\alpha + \beta + 2\gamma}{2} \sin \frac{\alpha + \beta}{2}\right) \quad (\text{since, } \sin (-A) = -\sin A) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \left(\cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta + 2\gamma}{2}\right)
 \end{aligned}$$

Again by using the formula,

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

$$\begin{aligned}
 &= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{\alpha - \beta}{2} + \frac{\alpha + \beta + 2\gamma}{2}}{2} \sin \frac{\frac{\alpha - \beta}{2} - \frac{\alpha + \beta + 2\gamma}{2}}{2} \right) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{\alpha - \beta + \alpha + \beta + 2\gamma}{2}}{2} \sin \frac{\frac{\alpha - \beta - (\alpha + \beta + 2\gamma)}{2}}{2} \right) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{2\alpha + 2\gamma}{2}}{2} \sin \frac{\frac{\alpha - \beta - \alpha - \beta - 2\gamma}{2}}{2} \right) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{2\alpha + 2\gamma}{2}}{2} \sin \frac{\frac{-2\beta - 2\gamma}{2}}{2} \right) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{2(\alpha + \gamma)}{2}}{2} \sin \frac{\frac{-2(\beta + \gamma)}{2}}{2} \right) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\alpha + \gamma}{2} \sin \frac{-(\beta + \gamma)}{2} \right) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \left(2 \sin \frac{\alpha + \gamma}{2} \sin \frac{\beta + \gamma}{2} \right) \\
 &= 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \sin \frac{\alpha + \gamma}{2}
 \end{aligned}$$

= RHS

Hence proved.

(ii) $\cos(A + B + C) + \cos(A - B + C) + \cos(A + B - C) + \cos(-A + B + C) = 4 \cos A \cos B \cos C$

Let us consider LHS:

$$\cos(A + B + C) + \cos(A - B + C) + \cos(A + B - C) + \cos(-A + B + C)$$

so,

$$\begin{aligned}
 &\cos(A + B + C) + \cos(A - B + C) + \cos(A + B - C) + \cos(-A + B + C) = \\
 &= \{\cos(A + B + C) + \cos(A - B + C)\} + \{\cos(A + B - C) + \cos(-A + B + C)\}
 \end{aligned}$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\begin{aligned}
 &= \left\{ 2 \cos \frac{(A + B + C) + (A - B + C)}{2} \cos \frac{(A + B + C) - (A - B + C)}{2} \right\} \\
 &+ \left\{ 2 \cos \frac{(A + B - C) + (-A + B + C)}{2} \cos \frac{(A + B - C) - (-A + B + C)}{2} \right\} \\
 &= \left\{ 2 \cos \frac{A + B + C + A - B + C}{2} \cos \frac{A + B + C - A + B - C}{2} \right\} \\
 &\quad + \left\{ 2 \cos \frac{A + B - C - A + B + C}{2} \cos \frac{A + B - C + A - B - C}{2} \right\} \\
 &= \left\{ 2 \cos \frac{2A + 2C}{2} \cos \frac{2B}{2} \right\} + \left\{ 2 \cos \frac{2B}{2} \cos \frac{2A - 2C}{2} \right\} \\
 &= \left\{ 2 \cos \frac{2(A + C)}{2} \cos \frac{2B}{2} \right\} + \left\{ 2 \cos \frac{2B}{2} \cos \frac{2(A - C)}{2} \right\} \\
 &= 2 \cos (A + C) \cos B + 2 \cos B \cos (A - C) \\
 &= 2 \cos B \{ \cos (A + C) + \cos (A - C) \}
 \end{aligned}$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$= 2 \cos B \left\{ 2 \cos \frac{A + C + (A - C)}{2} \cos \frac{A + C - (A - C)}{2} \right\}$$

$$= 2 \cos B \left\{ 2 \cos \frac{A + C + A - C}{2} \cos \frac{A + C - A + C}{2} \right\}$$

$$= 2 \cos B \left\{ 2 \cos \frac{2A}{2} \cos \frac{2C}{2} \right\}$$

$$= 4 \cos A \cos B \cos C$$

= RHS

Hence proved.