

EXERCISE 19.20

Evaluate the following integrals:

$$1. \int \frac{x^2 + x + 1}{x^2 - x} dx$$

Solution:

$$\text{Given } I = \int \frac{x^2 + x + 1}{x^2 - x} dx$$

$$\text{Expressing the integral } \int \frac{P(x)}{ax^2 + bx + c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2 + bx + c} dx$$

$$\Rightarrow \int \frac{x^2 + x + 1}{(x-1)x} dx$$

$$\Rightarrow \int \left(\frac{2x+1}{(x-1)x} + 1 \right) dx$$

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1 dx$$

$$\text{Consider } \int \frac{2x+1}{(x-1)x} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{2x+1}{(x-1)x} = \frac{A}{x-1} + \frac{B}{x}$$

$$\Rightarrow 2x+1 = Ax + B(x-1)$$

$$\Rightarrow 2x+1 = Ax + Bx - B$$

$$\Rightarrow 2x+1 = (A+B)x - B$$

$$\therefore B = -1 \text{ and } A + B = 2$$

$$\therefore A = 2 + 1 = 3$$

$$\text{Thus, } \Rightarrow \frac{2x+1}{(x-1)x} = \frac{3}{x-1} - \frac{1}{x}$$

$$\Rightarrow \int \left(\frac{3}{x-1} - \frac{1}{x} \right) dx$$

$$\Rightarrow 3 \int \frac{1}{x-1} dx - \int \frac{1}{x} dx$$

Consider $\int \frac{1}{x-1} dx$

Substitute $u = x - 1 \rightarrow dx = du$.

$$\Rightarrow \int \frac{1}{x-1} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\therefore \int \frac{1}{u} du = \log|u| = \log|x-1|$$

Then,

$$\Rightarrow 3 \int \frac{1}{x-1} dx - \int \frac{1}{x} dx = 3(\log|x-1|) - \int \frac{1}{x} dx$$

$$= 3(\log|x-1|) - \log|x|$$

$$\therefore \int \frac{2x+1}{(x-1)x} dx = 3(\log|x-1|) - \log|x|$$

Then,

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1 dx = 3(\log|x-1|) - \log|x| + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1 dx = 3(\log|x-1|) - \log|x| + x + c$$

$$\therefore I = \int \frac{x^2+x+1}{x^2-x} dx = -\log|x| + x + 3(\log|x-1|) + c$$

2. $\int \frac{x^2 + x - 1}{x^2 + x - 6} dx$

Solution:

Consider $I = \int \frac{x^2+x-1}{x^2+x-6} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

Let $x^2 + x - 1 = x^2 + x - 6 + 5$

$$\Rightarrow \int \frac{x^2 + x - 1}{x^2 + x - 6} dx = \int \left(\frac{x^2 + x - 6}{x^2 + x - 6} + \frac{5}{x^2 + x - 6} \right) dx$$

$$= \int \left(\frac{5}{x^2 + x - 6} + 1 \right) dx$$

$$= 5 \int \left(\frac{1}{x^2 + x - 6} \right) dx + \int 1 dx$$

Consider $\int \frac{1}{x^2+x-6} dx$

Factorizing the denominator,

$$\Rightarrow \int \frac{1}{x^2 + x - 6} dx = \int \frac{1}{(x-2)(x+3)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$$\Rightarrow 1 = A(x+3) + B(x-2)$$

$$\Rightarrow 1 = Ax + 3A + Bx - 2B$$

$$\Rightarrow 1 = (A+B)x + (3A-2B)$$

$$\Rightarrow \text{Then } A+B=0 \dots (1)$$

$$\text{And } 3A-2B=1 \dots (2)$$

Solving (1) and (2),

$$2 \times (1) \rightarrow 2A + 2B = 0$$

$$1 \times (2) \rightarrow 3A - 2B = 1$$

$$5A = 1$$

$$\therefore A = 1/5$$

Substituting A value in (1),

$$\Rightarrow A + B = 0$$

$$\Rightarrow 1/5 + B = 0$$

$$\therefore B = -1/5$$

$$\text{Thus, } \frac{1}{(x-2)(x+3)} = \frac{1}{5(x-2)} - \frac{1}{5(x+3)}$$

$$= \frac{1}{5} \int \frac{1}{x-2} dx - \frac{1}{5} \int \frac{1}{x+3} dx$$

Let $x - 2 = u \rightarrow dx = du$

And $x + 3 = v \rightarrow dx = dv$.

$$\Rightarrow \frac{1}{5} \int \frac{1}{u} du - \frac{1}{5} \int \frac{1}{v} dv$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \frac{1}{5} \log|u| - \frac{1}{5} \log|v|$$

$$\Rightarrow \frac{1}{5} \log|x-2| - \frac{1}{5} \log|x+3|$$

$$\Rightarrow \frac{1}{5} (\log|x-2| - \log|x+3|)$$

Then,

$$\Rightarrow 5 \int \left(\frac{1}{x^2+x-6} \right) dx + \int 1 dx = 5 \left(\frac{1}{5} (\log|x-2| - \log|x+3|) \right) + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\Rightarrow (\log|x-2| - \log|x+3|) + x + c$$

$$\therefore I = \int \frac{x^2+x-1}{x^2+x-6} dx = -\log|x+3| + x + \log|x-2| + c$$

Or $I = \log|(x-2)/(x+3)| + x + c$

$$3. \int \frac{(1-x^2)}{x(1-2x)} dx$$

Solution:

Given $I = \int \frac{1-x^2}{(1-2x)x} dx$

Rewriting, we get $\int \frac{x^2-1}{x(2x-1)} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)} dx = \int \left(\frac{x-2}{2x(2x-1)} + \frac{1}{2} \right) dx$$

$$= \frac{1}{2} \int \frac{x-2}{x(2x-1)} dx + \frac{1}{2} \int 1 dx$$

Consider $\int \frac{x-2}{x(2x-1)} dx$

By partial fraction decomposition,

$$\Rightarrow \frac{x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$$\Rightarrow x-2 = A(2x-1) + Bx$$

$$\Rightarrow x-2 = 2Ax - A + Bx$$

$$\Rightarrow x-2 = (2A+B)x - A$$

$$\therefore A = 2 \text{ and } 2A + B = 1$$

$$\therefore B = 1 - 4 = -3$$

Thus, $\Rightarrow \frac{x-2}{x(2x-1)} = \frac{2}{x} - \frac{3}{2x-1}$

$$\Rightarrow \int \left(\frac{2}{x} - \frac{3}{2x-1} \right) dx$$

$$\Rightarrow 2 \int \frac{1}{x} dx - 3 \int \frac{1}{2x-1} dx$$

Consider $\int \frac{1}{x} dx$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{x} dx = \log|x|$$

And consider $\int \frac{1}{2x-1} dx$

Let $u = 2x - 1 \rightarrow dx = 1/2 du$

$$\Rightarrow \int \frac{1}{2x-1} dx = \frac{1}{2} \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{u} du = \frac{\log|u|}{2} = \frac{\log|2x-1|}{2}$$

Then,

$$\Rightarrow \int \frac{x-2}{x(2x-1)} dx = 2 \int \frac{1}{x} dx - 3 \int \frac{1}{2x-1} dx$$

$$= 2(\log|x|) - 3 \left(\frac{\log|2x-1|}{2} \right)$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{x^2-1}{x(2x-1)} dx &= \frac{1}{2} \int \frac{x-2}{x(2x-1)} dx + \frac{1}{2} \int 1 dx \\ &= \frac{1}{2} \left(2(\log|x|) - 3 \left(\frac{\log|2x-1|}{2} \right) \right) + \frac{1}{2} \int 1 dx \end{aligned}$$

We know that $\int 1 dx = x + c$

$$\Rightarrow \log|x| - \frac{3 \log|2x-1|}{4} + \frac{x}{2} + c$$

$$\therefore I = \int \frac{1-x^2}{(1-2x)x} dx = -\frac{3 \log|2x-1|}{4} + \log|x| + \frac{x}{2} + c$$

$$4. \int \frac{x^2 + 1}{x^2 - 5x + 6} dx$$

Solution:

$$\text{Let } u = x^2 - 5x + 6 \rightarrow dx = \frac{1}{2x-5} du$$

$$\Rightarrow \int \frac{2x-5}{(x^2-5x+6)} dx = \int \frac{2x-5}{u} \frac{1}{2x-5} du$$

$$= \int \frac{1}{u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 - 5x + 6|$$

$$\text{Now consider } \int \frac{1}{x^2-5x+6} dx$$

$$\Rightarrow \int \frac{1}{x^2 - 5x + 6} dx = \int \frac{1}{(x-3)(x-2)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$\Rightarrow 1 = A(x-2) + B(x-3)$$

$$\Rightarrow 1 = Ax - 2A + Bx - 3B$$

$$\Rightarrow 1 = (A+B)x - (2A+3B)$$

$$\Rightarrow A+B=0 \text{ and } 2A+3B=-1$$

Solving the two equations,

$$\Rightarrow A+B=0$$

$$2A + 3B = -1$$

$$-B = 1$$

$$\therefore B = -1 \text{ and } A = 1$$

$$\begin{aligned} \Rightarrow \int \frac{1}{(x-3)(x-2)} dx &= \int \left(\frac{1}{x-3} - \frac{1}{x-2} \right) dx \\ &= \int \frac{1}{x-3} dx - \int \frac{1}{x-2} dx \end{aligned}$$

Consider $\int \frac{1}{x-3} dx$

Let $u = x - 3 \rightarrow dx = du$

$$\Rightarrow \int \frac{1}{x-3} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x-3|$$

Similarly $\int \frac{1}{x-2} dx$

Let $u = x - 2 \rightarrow dx = du$

$$\Rightarrow \int \frac{1}{x-2} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x-2|$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{1}{x^2 - 5x + 6} dx &= \int \frac{1}{(x-3)(x-2)} dx = \int \frac{1}{x-3} dx - \int \frac{1}{x-2} dx \\ &= \log|x-3| - \log|x-2| \end{aligned}$$

$$\begin{aligned} \Rightarrow \int \frac{x-1}{x^2-5x+6} dx &= \frac{1}{2} \int \frac{2x-5}{(x^2-5x+6)} dx + \frac{3}{2} \int \frac{1}{x^2-5x+6} dx \\ &= \frac{1}{2} (\log|x^2-5x+6|) + \frac{3}{2} (\log|x-3| - \log|x-2|) \\ &= \frac{\log|x^2-5x+6|}{2} + \frac{3\log|x-3|}{2} - \frac{3\log|x-2|}{2} \end{aligned}$$

Then,

$$\Rightarrow \int \frac{x^2+1}{x^2-5x+6} dx = 5 \int \frac{x-1}{x^2-5x+6} dx + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\begin{aligned} \Rightarrow 5 \int \frac{x-1}{x^2-5x+6} dx + \int 1 dx \\ &= \frac{5\log|x^2-5x+6|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c \\ &= \frac{5\log|x-2|\log|x-3|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c \\ &= x - 5\log|x-2| + 10\log|x-3| + c \end{aligned}$$

$$\therefore I = \int \frac{x^2+1}{x^2-5x+6} dx = x - 5\log|x-2| + 10\log|x-3| + c$$

5. $\int \frac{x^2}{x^2+7x+10} dx$

Solution:

Given $I = \int \frac{x^2}{x^2+7x+10} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2}{x^2+7x+10} dx = \int \left(\frac{-7x-10}{x^2+7x+10} + 1 \right) dx$$

$$= - \int \frac{7x + 10}{x^2 + 7x + 10} dx + \int 1 dx$$

Consider $\int \frac{7x+10}{x^2+7x+10} dx$

Let $7x + 10 = \frac{7}{2}(2x + 7) - \frac{29}{2}$ and split,

$$\Rightarrow \int \frac{7x + 10}{x^2 + 7x + 10} dx = \int \left(\frac{7(2x + 7)}{2(x^2 + 7x + 10)} - \frac{29}{2(x^2 + 7x + 10)} \right) dx$$

$$= \frac{7}{2} \int \frac{2x + 7}{x^2 + 7x + 10} dx - \frac{29}{2} \int \frac{1}{x^2 + 7x + 10} dx$$

Consider $\int \frac{2x+7}{x^2+7x+10} dx$

Let $u = x^2 + 7x + 10 \rightarrow dx = \frac{1}{2x+7} du$

$$\Rightarrow \int \frac{2x + 7}{(x^2 + 7x + 10)} dx = \int \frac{2x + 7}{u} \cdot \frac{1}{2x + 7} du$$

$$= \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 + 7x + 10|$$

Now consider $\int \frac{1}{x^2+7x+10} dx$

$$\Rightarrow \int \frac{1}{x^2 + 7x + 10} dx = \int \frac{1}{(x + 2)(x + 5)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x + 2)(x + 5)} = \frac{A}{x + 2} + \frac{B}{x + 5}$$

$$\Rightarrow 1 = A(x + 2) + B(x + 5)$$

$$\Rightarrow 1 = Ax + 2A + Bx + 5B$$

$$\Rightarrow A + B = 0 \text{ and } 2A + 5B = 1$$

Solving the two equations,

$$\Rightarrow 2A + 2B = 0$$

$$2A + 5B = 1$$

$$-3B = -1$$

$$\therefore B = 1/3 \text{ and } A = -1/3$$

$$\begin{aligned} \Rightarrow \int \frac{1}{(x+2)(x+5)} dx &= \int \left(\frac{-1}{3(x+2)} + \frac{1}{3(x+5)} \right) dx \\ &= -\frac{1}{3} \int \frac{1}{x+2} dx + \frac{1}{3} \int \frac{1}{x+5} dx \end{aligned}$$

Consider $\int \frac{1}{x+2} dx$

$$\text{Let } u = x + 2 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{x+2} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x + 2|$$

Similarly $\int \frac{1}{x+5} dx$

$$\text{Let } u = x + 5 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{x+5} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x + 5|$$



Then,

$$\begin{aligned} \Rightarrow \int \frac{1}{x^2 + 7x + 10} dx &= \int \frac{1}{(x+2)(x+5)} dx = -\frac{1}{3} \int \frac{1}{x+2} dx + \frac{1}{3} \int \frac{1}{x+5} dx \\ &= \frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3} \end{aligned}$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{7x+10}{x^2+7x+10} dx &= \frac{7}{2} \int \frac{2x+7}{x^2+7x+10} dx - \frac{29}{2} \int \frac{1}{x^2+7x+10} dx \\ &= \frac{7}{2} (\log|x^2+7x+10|) - \frac{29}{2} \left(\frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3} \right) \\ &= \frac{7 \log|x^2+7x+10|}{2} + \frac{29 \log|x+2|}{6} - \frac{29 \log|x+5|}{6} \end{aligned}$$

Then,

$$\Rightarrow \int \frac{x^2}{x^2+7x+10} dx = - \int \frac{7x+10}{x^2+7x+10} dx + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\begin{aligned} \Rightarrow - \int \frac{7x+10}{x^2+7x+10} dx + \int 1 dx \\ = \frac{-7 \log|x^2+7x+10|}{2} - \frac{29 \log|x+2|}{6} + \frac{29 \log|x+5|}{6} + x + c \end{aligned}$$

Hence,

$$I = x - \frac{7}{2} \log|x^2+7x+10| + \frac{29}{6} \log \left| \frac{x+2}{x+5} \right| + c$$